ON PROPER SPLINTERS IN POSITIVE CHARACTERISTIC

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Abstract. While the splinter property is a local property for Noetherian schemes in characteristic zero, Bhatt observed that it imposes strong conditions on the global geometry of proper schemes in positive characteristic. We show that if a proper scheme over a field of positive characteristic is a splinter, then its Nori fundamental group is trivial and its Kodaira dimension is negative. In another direction, Bhatt also showed that any splinter in positive characteristic is a derived splinter. We ask whether the splinter property is a derived-invariant for projective varieties in positive characteristic and give a positive answer for Gorenstein projective varieties with big anticanonical divisor. For that purpose, we introduce the notion of $\mathcal{O}$-equivalence and show that the derived splinter property for schemes of finite type and separated over a fixed Noetherian base is preserved under $\mathcal{O}$-equivalence. Finally, we show that global F-regularity is a derived-invariant for normal Gorenstein projective varieties in positive characteristic.

1. Introduction

A Noetherian scheme X is a *splinter* if for all finite surjective morphisms $f\colon Z \to X$ the map $\mathcal{O}_X \to f_*\mathcal{O}_Z$ splits in the category of coherent $\mathcal{O}_X$-modules. The direct summand conjecture, now a theorem due to André [And18], stipulates that any regular Noetherian ring is a splinter. In characteristic zero, the splinter property is a local property : a Noetherian scheme over $\mathbb{Q}$ is a splinter if and only if it is normal. In positive characteristic, the splinter property is no longer a local property in general. Bhatt's beautiful [Bha12, Thm. 1.5], inspired from Hochster and Huneke's [HH92, Thm. 1.2], shows that, for a proper scheme over an affine Noetherian scheme of positive characteristic, the positive-degree cohomology of the structure sheaf vanishes up to finite covers. Bhatt draws two consequences for splinters of positive characteristic : first that the positive-degree cohomology of semiample line bundles on proper splinters vanishes, and second that splinters and derived splinters coincide. Our first aim is to provide further global constraints on proper splinters in positive characteristic. Our second aim is to study whether the splinter property is a derived-invariant for projective varieties in positive characteristic.

Global constraints on proper splinters in positive characteristic. Recall that a smooth projective, separably rationally connected, variety over an algebraically closed field of positive characteristic has trivial Nori fundamental group [Bis09], has negative Kodaira dimension [Kol96, Ch. IV, Cor. 1.11 & Prop. 3.3], and has no nonzero global differential forms [Kol96, Ch. IV, Cor. 3.8]. Motivated by the intriguing question whether proper splinters over an algebraically closed field of positive characteristic are separably rationally connected, we show :

Theorem. *Let X be a connected proper scheme over a field k of positive characteristic. Assume that X is a splinter.*

- **(A)** (Theorem 7.9) *If X has a k-rational point $x \in X(k)$, then the Nori fundamental group $\pi_1^N(X,x)$ is trivial. In particular, if k is algebraically closed, then any finite torsor over X is trivial.*
- **(B)** (Theorem 8.1) *If X is positive-dimensional, then X has negative Kodaira dimension, i.e., $H^0(X, \mathcal{O}_X(nK_X)) = 0$ for all $n > 0$, where K_X denotes the canonical divisor of X.*
- **(C)** (Theorem 9.2) *If X is smooth, then $H^0(X, \Omega^1_X) = 0$.*

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Note from their respective proofs that **(C)**, and **(B)** in the Gorenstein case, follow from the known fact, recalled in Proposition 8.4, that the Picard group of a proper splinter over a field of positive characteristic is torsion-free. The proofs of **(A)**, and of **(B)** in the general non-Gorenstein case, rely on a lifting property for splinters along torsors established in Proposition 7.4, which itself relies on the more general lifting property established in Lemma 5.3. The new idea, which makes it in particular possible to avoid any Gorenstein assumption, is the use of the exceptional inverse image functor for finite morphisms. As an aside, by using the more general exceptional inverse image functor for proper morphisms, we further observe in Proposition 5.4 that the splinter property for Noetherian schemes of positive characteristic lifts along crepant morphisms. We establish the following lifting property for splinters along finite quasi-torsor morphisms :

Proposition (Proposition 7.5(ii))**.** *Let $\pi\colon Y \to X$ be a morphism of normal Noetherian Nagata schemes over a Noetherian ring R such that either $H^0(Y, \mathcal{O}_Y)$ is a field or $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$. Assume that π is a finite quasi-torsor, i.e., that there exists a Zariski open subset $U \subseteq X$ with $\operatorname{codim}_X(X \setminus U) \geq 2$ such that $\pi^{-1}(U) \to U$ is a torsor under a finite group scheme over R. If X is a splinter, then Y is a splinter.*

In fact, Proposition 7.5(ii) is stated more generally (by Remark 3.4) for globally $+$-regular pairs as introduced in [Bha+22], and extends [Bha+22, Prop. 6.20] which deals with the quasi-étale case. Regarding Theorem **(A)**, we also show in Theorem 7.9 that if X is a proper splinter over a separably closed field of positive characteristic, then its étale fundamental group is trivial. In that direction, we also refer to [Cai+23, Thm. 7.0.3], where it is in particular showed that the étale fundamental group of the regular locus of a normal projective globally $+$-regular variety satisfying some additional technical assumptions is finite, but also to the references in the introduction of *loc. cit.* regarding the étale fundamental group of regular loci of near smooth Fano varieties.

For proper surfaces, we have the following results regarding splinters. Let k be an algebraically closed field of positive characteristic. It is well-known that a proper curve over k is a splinter if and only if it is isomorphic to $\mathbb{P}^1_k$. In Proposition 10.1, we use Theorem 8.1 to show that if a proper surface over k is a splinter, then it is rational. We provide in Proposition 10.4 new examples of proper rational surfaces that are splinters, by establishing that the blow-up of $\mathbb{P}^2_k$ in any number of points lying on a conic is a splinter. On the other hand, in Proposition 10.10 and Proposition 10.12, we give examples of proper rational surfaces that are not splinters. For instance, we show that over a finite field the blow-up of $\mathbb{P}^2_k$ in 9 points lying on a smooth cubic curve is not a splinter.

$\mathcal{O}$-invariance and D-invariance of the splinter property. The second aim of this paper is to study whether the splinter property, and the related notion of *global F-regularity*, is a derived-invariant among projective varieties. We say that two projective varieties X and Y over a field k are D-equivalent if there is a k-linear equivalence $\mathsf{D}^b(X) \cong \mathsf{D}^b(Y)$ between their bounded derived categories of coherent sheaves. Given that a Gorenstein projective splinter, resp. a Gorenstein projective globally F-regular variety, in positive characteristic is expected to, resp. is known to, have big anticanonical divisor (see Conjecture 3.12 due to [Bha+22], resp. Proposition 3.11 due to [SS10, Cor. 4.5]), we obtain the following positive answer :

Theorem. *Let X and Y be normal Gorenstein projective varieties of a field k of positive characteristic. Assume that X and Y are D-equivalent. Then :*

- **(D)** (Corollary 11.18) *X is a splinter if and only if Y is a splinter, provided $-K_X$ is big.*
- **(E)** (Corollary 11.22) *X is globally F-regular if and only if Y is globally F-regular, provided k is F-finite.*

For that purpose, we introduce in Definition 11.6 the notion of $\mathcal{O}$-equivalence for separated schemes of finite type over a Noetherian base. By Proposition 11.7, this notion coincides with the classical notion of K-equivalence in the case of normal Gorenstein varieties over a field. As before, but now for proper morphisms that are not necessarily finite, the new idea is to use the exceptional inverse image functor of Grothendieck, which allows for more flexibility. In Proposition 11.12, we observe that Kawamata's [Kaw02, Thm. 1.4(2)], stating that two D-equivalent smooth projective complex varieties X and Y with K_X or $-K_X$ big are K-equivalent, extends to the case of normal

Gorenstein projective varieties over an arbitrary field (although it might not be necessary, this is the only place where the Gorenstein assumption is needed). As such, Theorem **(D)** and Theorem **(E)** follow from the following:

Theorem. *Let X and Y be varieties of a field k of positive characteristic. Assume that X and Y are $\mathcal{O}$-equivalent. Then:*

(F) (Theorem 11.15) *X is a splinter if and only if Y is a splinter.*

Assume in addition that X and Y are normal quasi-projective and that k is F-finite. Then:

(G) (Theorem 11.20) *X is globally F-regular if and only if Y is globally F-regular.*

More generally, we show in Theorem 11.15 that the *derived splinter* property is invariant under $\mathcal{O}$-equivalence for integral schemes of finite type and separated over a Noetherian scheme S. We observe that both Theorems 11.15 and 11.20 hold without any ($\mathbb{Q}$-)Gorenstein assumption and in fact without any restrictions on the singularities of X nor Y. Again, this is made possible by the systematic use of the exceptional inverse image functor. For the sake of illustration, we explain in Section 11.1 how Theorems 11.15 and 11.20 follow in the case of normal *terminal* varieties over k from the fact that a K-equivalence between two such varieties induces a small birational map. Note however that in positive characteristic, splinters and globally F-regular varieties may have worse singularities; in fact, both the splinter property and global F-regularity are expected to locally be the analogues of klt singularities in the complex setting.

Organization of the paper. In Sections 2 to 4, we mostly fix notation and collect basic and known facts about splinters and globally F-regular varieties. Our first new contributions are contained in Sections 5 and 6 dealing respectively with splinters and globally F-regular varieties. Notably, the use of the exceptional inverse image functor makes its first appearance in Section 5.2, where we observe that the splinter property lifts along finite surjective morphisms $\pi\colon Y \to X$ of Noetherian schemes such that $\pi^!\mathcal{O}_X \cong \mathcal{O}_Y$, under the condition that either $H^0(Y, \mathcal{O}_Y)$ is a field or $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$.

In Sections 7 to 10, we explore global constraints on proper splinters in positive characteristic and establish Theorems **(A)** to **(C)**. In Section 10, we show that proper splinter surfaces in positive characteristic are rational, and give examples of rational surfaces that are splinters as well as examples of rational surfaces that are not splinters.

Finally in Section 11, which can be read mostly independently of the rest of the paper, we introduce the notion of $\mathcal{O}$-equivalence for separated schemes of finite type over a Noetherian scheme S and compare it, in the case of Gorenstein projective varieties over a field, to the usual notions of K-equivalence and D-equivalence. We then establish Theorems **(D)** to **(G)**.

Conventions. A *variety* is an integral separated scheme of finite type over a field. For a scheme X over the finite field $\mathbb{F}_p$ with p elements, the *Frobenius* is denoted by $F\colon X \to X$; it is the identity on the underlying topological space and sends each local section of $\mathcal{O}_X$ to its p-th power. A scheme X over $\mathbb{F}_p$ is said to be *F-finite* if the Frobenius map $F\colon X \to X$ is finite. Note that a variety over an F-finite field is an F-finite scheme. For a scheme X, we denote by X_{reg} its regular locus and by $X_{\mathrm{sing}} \coloneqq X \setminus X_{\mathrm{reg}}$ its singular locus; if X is J-1, e.g. if X is excellent, then $X_{\mathrm{reg}} \subseteq X$ is an open embedding [Sta23, Tag 07P6]. We denote by $\nu\colon X^\nu \to X$ the normalization of X; if X is Nagata, e.g. if X is excellent, then ν is a finite morphism [Sta23, Tag 035S]. Both the Nagata and the excellent properties are stable under locally of finite type extensions; see [Sta23, Tag 0359] and [Sta23, Tag 07QS]. For a finite, resp. proper, morphism $f\colon Z \to X$, we denote by $\eta_f\colon \mathcal{O}_X \to f_*\mathcal{O}_Z$, resp. $\eta_f\colon \mathcal{O}_X \to \mathrm{R}f_*\mathcal{O}_Z$, the canonical morphism in the category, resp. derived category, of coherent sheaves on X. Given a Weil divisor D on a normal scheme X, we write $\sigma_D\colon \mathcal{O}_X \to \mathcal{O}_X(D)$ for the morphism determined by D.

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2. Reflexive sheaves and dualizing complexes

2.1. **Reflexive sheaves and Weil divisors.** Let X be an integral Noetherian scheme. Recall that a coherent sheaf $\mathcal{F}$ on X is called *reflexive* if the canonical map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism, where by definition $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. The following facts can be found for example in [Sta23, Tag 0AVT]:

(i) For any coherent sheaf $\mathcal{F}$ and any reflexive sheaf $\mathcal{G}$ the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is reflexive.
(ii) If X is normal, then a sheaf is reflexive if and only if it is S_2.
(iii) If X is normal and $i\colon U \hookrightarrow X$ is an open immersion such that $\operatorname{codim}_X(X \setminus U) \geq 2$, then $i_*i^*\mathcal{F} \cong \mathcal{F}$ for any reflexive sheaf $\mathcal{F}$. Furthermore, the restriction i^* induces an equivalence of categories from reflexive coherent sheaves on X to reflexive coherent sheaves on U.

To any Weil divisor D on an integral normal Noetherian scheme X with function field $K(X)$, one can associate a coherent sheaf $\mathcal{O}_X(D) \subseteq K(X)$ whose sections on open subsets $V \subseteq X$ are given by

$$\Gamma(V, \mathcal{O}_X(D)) := \{f \in K(X)^\times \mid \operatorname{div}(f)|_V + D|_V \geq 0\} \cup \{0\}.$$

The sheaf $\mathcal{O}_X(D)$ is reflexive of rank 1 and it is a line bundle if and only if D is Cartier. Since any reflexive rank 1 sheaf is a subsheaf of the locally constant sheaf $K(X)$, we have [Sta23, Tag 0EBM] a 1-1 correspondence

$$\{\text{Weil divisors on } X \text{ up to linear equivalence}\} \leftrightarrow \{\text{reflexive sheaves of rank 1 on } X\}/\cong.$$

Moreover thanks to (iii) and to the fact that Weil divisors on regular schemes are Cartier, if X is in addition assumed to be excellent (in which case, the regular locus of X is open and, by normality, dense in X), then this bijection turns out to be a group homomorphism provided one takes the double dual of the usual tensor product on the right hand side, i.e.,

$$\mathcal{O}_X(D + D') \cong (\mathcal{O}_X(D) \otimes \mathcal{O}_X(D'))^{\vee\vee}.$$

A Weil divisor D on a normal Noetherian scheme is effective if and only if $\mathcal{O}_X \subseteq \mathcal{O}_X(D) \subseteq K(X)$. Thus a reflexive rank 1 sheaf $\mathcal{F}$ corresponds to an effective Weil divisor D if and only if there is an injective morphism $\mathcal{O}_X \hookrightarrow \mathcal{F}$. For later use recall the following criterion.

Lemma 2.1. *Let X be an integral normal Noetherian scheme and assume that $H^0(X, \mathcal{O}_X)$ is a field. Then a Weil divisor D on X is linearly equivalent to 0 if and only if both $\mathcal{O}_X(D)$ and $\mathcal{O}_X(-D)$ admit nonzero global sections.*

Proof. The only if part is obvious. Assume $s\colon \mathcal{O}_X \to \mathcal{O}_X(D)$ and $t\colon \mathcal{O}_X \to \mathcal{O}_X(-D)$ are nontrivial sections. We have isomorphisms of sheaves $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X(-D)) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X)$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X(D)) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$. Therefore we can interpret t as a global section of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X)$. The compositions $s \circ t$ and $t \circ s$ are both nonzero and give elements in $H^0(X, \mathcal{O}_X)$, thus they are isomorphisms. Hence $\mathcal{O}_X \cong \mathcal{O}_X(D)$, which shows that D is trivial. □

Given two Weil divisors D and E on a normal Noetherian scheme X, we write $D \sim E$ if D and E are linearly equivalent. Likewise, for two $\mathbb{Q}$-Weil divisors D and E, we write $D \sim_{\mathbb{Q}} E$ if D and E are $\mathbb{Q}$-linearly equivalent.

Let $f\colon Y \to X$ be a finite surjective morphism between normal excellent Noetherian schemes and let D be a $\mathbb{Q}$-Weil divisor on X. Then, see, e.g., [KM98, Proof of Prop. 5.20], the pullback f^*D can be defined by restricting D to X_{reg}, pulling back to $f^{-1}(X_{\mathrm{reg}})$, and then extending the pullback uniquely to a Weil divisor on Y, which is possible since $\operatorname{codim}_Y(Y \setminus f^{-1}(X_{\mathrm{reg}})) \geq 2$.

2.2. **Exceptional inverse image functor and dualizing complexes.** Let $h\colon X \to S$ be a scheme of finite type and separated over a Noetherian scheme S. The *exceptional inverse image functor* $h^!$ is well-defined, see, e.g., [Sta23, Tag 0AU3], and if S admits a dualizing complex $\omega_S^\bullet$, then $\omega_X^\bullet := h^!\omega_S^\bullet$ is a dualizing complex on X. More generally, if $f\colon X \to Y$ is an S-morphism of schemes of finite type and separated over a Noetherian scheme, the exceptional inverse image functor $f^!$ is well-defined, and if f is proper it is right adjoint to $\mathrm{R}f_*$. If X is equidimensional, $\omega_X := \mathcal{H}^{-\dim X}(\omega_X^\bullet)$ is an S_2 sheaf, called the *dualizing sheaf* [Sta23, Tag 0AWH]. Moreover, X is Cohen–Macaulay if and only if $\omega_X^\bullet = \omega_X[\dim X]$ [Sta23, Tag 0AWQ] and X is Gorenstein if and only if $\omega_X^\bullet$ is an invertible object [Sta23, Tag 0AWV]. The latter condition is further equivalent to $\omega_X^\bullet = \omega_X[\dim X]$ with ω_X an invertible sheaf [Sta23, Tag 0FPG]. If X is normal and equidimensional, then ω_X is a reflexive sheaf of rank 1 ; thus, there exists a unique (up to linear equivalence) Weil divisor K_X, called the *canonical divisor*, such that $\omega_X = \mathcal{O}_X(K_X)$.

Let now $h\colon X \to \operatorname{Spec} k$ be a scheme of finite type and separated over a field k. Then $\omega_X^\bullet := h^!\mathcal{O}_{\operatorname{Spec} k} \in \mathsf{D}^b(X)$ is a dualizing complex on X. Here and throughout this paper $\mathsf{D}^b(X) := \mathsf{D}^b(\operatorname{Coh}(X))$ is the bounded derived category of coherent $\mathcal{O}_X$-modules. If X is smooth over k, then the dualizing sheaf ω_X coincides with the *canonical sheaf* $\bigwedge^{\dim X}\Omega^1_{X/k}$ [Sta23, Tag 0E9Z]. If $h\colon X \to \operatorname{Spec} k$ is proper and X is equidimensional, then $\mathrm{R}h_*$ is left adjoint to $h^!$ and for every $K \in \mathsf{D}^b(X)$, there is a functorial isomorphism $\operatorname{Ext}^i_X(K, \omega_X^\bullet) = \operatorname{Hom}_k(H^i(X,K), k)$ compatible with shifts and exact triangles, see, e.g., [Sta23, Tag 0FVU]. By Yoneda, the object $\omega_X^\bullet$ is unique up to unique isomorphism among all objects satisfying this universal property.

The following general Lemma 2.2 is a consequence of the properties of the exceptional inverse image functor. In particular, choosing $q\colon Z \to Y$ in the statement of Lemma 2.2 to be an isomorphism, it shows that if $p\colon Z \to X$ is a separated morphism of finite type, then $p^!\mathcal{O}_X \cong \mathcal{O}_Z$ if and only if $\mathrm{L}p^*\omega_X^\bullet \cong \omega_Z^\bullet$.

Lemma 2.2. *Let X and Y be schemes of finite type and separated over a Noetherian scheme S such that S admits a dualizing complex $\omega_S^\bullet$. Denote by $h_X\colon X \to S$ and $h_Y\colon Y \to S$ the structure morphisms. Let Z be a scheme of finite type and separated over S with S-morphisms $p\colon Z \to X$ and $q\colon Z \to Y$. Then in $\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_Z)$ we have*

$$\mathrm{L}p^*\omega_X^\bullet \cong \mathrm{L}q^*\omega_Y^\bullet \iff p^!\mathcal{O}_X \cong q^!\mathcal{O}_Y,$$

where $\omega_X^\bullet = h_X^!\omega_S^\bullet$ and $\omega_Y^\bullet = h_Y^!\omega_S^\bullet$. In particular, if X and Y are equidimensional and Gorenstein, then

$$p^*\omega_X[\dim X] \cong q^*\omega_Y[\dim Y] \iff p^!\mathcal{O}_X \cong q^!\mathcal{O}_Y,$$

where $\omega_X = \mathcal{H}^{-\dim X}(\omega_X^\bullet)$ and $\omega_Y = \mathcal{H}^{-\dim Y}(\omega_Y^\bullet)$.

Proof. Let $\omega_Z^\bullet := p^!\omega_X^\bullet = q^!\omega_Y^\bullet$ and recall, e.g. from [Sta23, Tag 0AU3], that the functor $\mathrm{R}\mathcal{H}om_{\mathcal{O}_Z}(-,\omega_Z^\bullet)$ defines an involution of $\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_Z)$. Moreover, the formula

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_Z}(p^!M, \omega_Z^\bullet) = \mathrm{L}p^*\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(M, \omega_X^\bullet)$$

holds naturally in $M \in \mathsf{D}^+_{\mathrm{Coh}}(\mathcal{O}_X)$ and similarly for $q\colon Z \to Y$. Setting $M = \mathcal{O}_X$ or $M = \mathcal{O}_Y$ yields

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_Z}(p^!\mathcal{O}_X, \omega_Z^\bullet) = \mathrm{L}p^*\omega_X^\bullet \quad \text{and} \quad \mathrm{R}\mathcal{H}om_{\mathcal{O}_Z}(q^!\mathcal{O}_Y, \omega_Z^\bullet) = \mathrm{L}q^*\omega_Y^\bullet.$$

Therefore, $p^!\mathcal{O}_X \cong q^!\mathcal{O}_Y$ if and only if $\mathrm{L}p^*\omega_X^\bullet \cong \mathrm{L}q^*\omega_Y^\bullet$. □

Remark 2.3. Let $\pi\colon Y \to X$ is a separated morphism of finite type of Noetherian schemes. If X admits a dualizing complex $\omega_X^\bullet$ and $\pi^!\mathcal{O}_X \cong \mathcal{O}_Y$, then Lemma 2.2 implies $\mathrm{L}p^*\omega_X^\bullet \cong \omega_Y^\bullet$. Since a scheme admitting a dualizing complex is Gorenstein if and only if it admits an invertible dualizing complex, we observe that if X is Gorenstein, then Y is Gorenstein. Likewise, since a scheme admitting a dualizing complex is Cohen–Macaulay if and only if it admits a dualizing complex that is the shift of a sheaf, we observe that if X is Cohen–Macaulay, then Y is Cohen–Macaulay.

Remark 2.4. A proper surjective morphism of normal excellent Noetherian schemes $\pi\colon Y \to X$ such that $\pi^!\mathcal{O}_X \cong \mathcal{O}_Y$ and such that X admits a dualizing complex is generically finite. Indeed, the regular locus X_{reg} of X is open (since X is excellent) and dense (since X is normal) and, likewise, the regular locus U of $\pi^{-1}(X_{\mathrm{reg}})$ is open and dense. Denote by $p\colon U \to X_{\mathrm{reg}}$ the restriction of π. Since the restriction to open subsets commutes with exceptional inverse image functors [Sta23, Tag 0G4J], we have $p^!\mathcal{O}_{X_{\mathrm{reg}}} \cong \mathcal{O}_U$. By Lemma 2.2, we have an isomorphism $\omega_U[\dim Y] \cong p^*\omega_{X_{\mathrm{reg}}}[\dim X]$, and thus $\dim Y = \dim X$.

3. Preliminaries on splinters and globally F-regular varieties

3.1. **Splinters.** We review the notion of *splinter* for Noetherian schemes, the local constraints it imposes, as well as the global constraints it imposes on proper schemes over a field of positive characteristic.

Definition 3.1. A Noetherian scheme X is a *splinter* if for any finite surjective morphism $f\colon Y \to X$ the canonical map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ splits in the category $\mathrm{Coh}(X)$ of coherent sheaves on X. A Noetherian scheme X is a *derived splinter* if for any proper surjective morphism $f\colon Y \to X$ the map $\mathcal{O}_X \to \mathrm{R}f_*\mathcal{O}_Y$ splits in the bounded derived category $\mathsf{D}^b(X)$ of coherent sheaves on X.

Note that a derived splinter is a splinter, so that being a derived splinter is *a priori* more restrictive than being a splinter. In the recent work [Bha+22], the notion of splinter has been extended to pairs. Precisely :

Definition 3.2 ([Bha+22, Def. 6.1])**.** Let (X, Δ) be a pair consisting of a normal excellent Noetherian scheme X and of an effective $\mathbb{Q}$-Weil divisor Δ. The pair (X, Δ) is called *globally $+$-regular* if for any finite surjective morphism $f\colon Y \to X$ with Y normal, the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lfloor f^*\Delta \rfloor)$ splits in $\mathrm{Coh}(X)$.

Remark 3.3. Note that [Bha+22, Def. 6.1] only consider schemes whose closed points have residue fields of positive characteristic. A reason for this is outlined in [Bha+22, Rmk. 6.3]. Since for our applications this condition is not necessary, we deviate from that convention.

Remark 3.4. It is obvious that if a normal excellent Noetherian scheme X is a splinter, then $(X, 0)$ is globally $+$-regular. For the converse, recall from [Sta23, Tag 035S] that the normalization of an excellent scheme is a finite morphism. Thus, if $\pi\colon Y \to X$ is a finite surjective morphism, then Y is excellent so that the normalization $\nu\colon Y^\nu \to Y$ is finite. Any splitting of $\mathcal{O}_X \to (\pi \circ \nu)_*\mathcal{O}_{Y^\nu}$ provides a splitting of $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$.

In general, if a Noetherian scheme X is a splinter, then it is a basic fact that any open $U \subseteq X$ is a splinter ; see, e.g., Lemma 4.1(i) below. Hence, if X is a splinter, then all of its local rings are splinters. Moreover, if X is in addition assumed to be affine, X is a splinter if and only if all its local rings are splinters [DT23, Lem. 2.1.3].

In characteristic zero, a Noetherian scheme X is a splinter if and only if it is normal [Bha12, Ex. 2.1], and it is a derived splinter if and only if it has rational singularities [Bha12, Thm. 2.12]. In particular, in characteristic zero, the splinter and derived splinter properties are distinct, and they both define local properties.

In positive characteristic, Bhatt showed that the splinter and the derived splinter properties agree [Bha12, Thm. 1.4] and observed that, in contrast to the affine setting, the splinter property is not a local property for proper schemes. The following proposition summarizes the known local constraints on splinters in positive characteristic.

Proposition 3.5 ([Bha12, Ex. 2.1, Rmk. 2.5, Cor. 6.4], [Bha21, Rmk. 5.14], [Sin99], [Smi00, §2.2])**.** *Let X be a scheme of finite type over a field of positive characteristic. If X is a splinter, then*

(i) *X is normal ;*
(ii) *X is Cohen–Macaulay ;*
(iii) *X is pseudo-rational ;*
(iv) *X is F-rational.*

Moreover, if X is $\mathbb{Q}$-Gorenstein, then X is F-regular, that is, if k is assumed to be F-finite, its local rings are strongly F-regular.

Recall from [HH89] that an F-finite ring R of positive characteristic is *strongly F-regular* if for any $c \in R$ not belonging to any minimal prime ideal of R, there exists $e > 0$ such that the inclusion of R-modules $R \hookrightarrow F^e_*R$ which sends 1 to F^e_*c splits as a map of R-modules. The ring R is strongly F-regular if and only if its local rings are strongly F-regular. If R is strongly F-regular, then the affine scheme $X = \operatorname{Spec} R$ is a splinter ; see, e.g., [MP19, Thm. 4.8].

The splinter property also imposes strong constraints on the global geometry of proper varieties in positive characteristic. For example, from Bhatt's "vanishing up to finite cover in positive characteristic" [Bha12, Thm. 1.5], we have :

Proposition 3.6 ([Bha12])**.** *Let X be a proper variety over a field of positive characteristic and let $\mathcal{L}$ be a semiample line bundle on X. If X is a splinter, then $H^i(X, \mathcal{L}) = 0$ for all $i > 0$. In particular, $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.*

Proof. For a proper variety X over a field of positive characteristic, there exists, by [Bha12, Prop. 7.2], for any $i > 0$ a finite surjective morphism $\pi\colon Y \to X$ such that the induced map $H^i(X, \mathcal{L}) \to H^i(Y, \pi^*\mathcal{L})$ is zero. If now X is a splinter, the pullback map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ admits a splitting s, i.e., we have

$$\mathrm{id}\colon \mathcal{O}_X \to \pi_*\mathcal{O}_Y \xrightarrow{s} \mathcal{O}_X.$$

Tensoring with $\mathcal{L}$ and using the projection formula, we obtain

$$\mathrm{id}\colon H^i(X, \mathcal{L}) \xrightarrow{0} H^i(Y, \pi^*\mathcal{L}) = H^i(X, \pi_*\pi^*\mathcal{L}) \to H^i(X, \mathcal{L}),$$

where the equality in the middle uses that π is finite, in particular affine. We conclude that $H^i(X, \mathcal{L}) = 0$. □

As a direct consequence, we have the following useful constraint, which will be refined in Theorem 8.1, on proper splinters in positive characteristic :

Lemma 3.7. *Let X be a proper scheme over a field of positive characteristic, with positive-dimensional irreducible components. If X is a splinter, then its canonical divisor K_X is not effective, in particular its dualizing sheaf ω_X is nontrivial.*

Proof. Since a splinter is normal, by working on each connected component of X separately, we can assume that X is of pure positive dimension, say n. By Proposition 3.5, X is Cohen–Macaulay, so $\omega_X^\bullet = \omega_X[n]$. By Proposition 3.6 and Serre duality for Cohen–Macaulay schemes, we obtain $H^0(X, \omega_X)^\vee \cong H^n(X, \mathcal{O}_X) = 0$. This shows that the Weil divisor K_X is not effective. □

3.2. **Globally F-regular varieties.** Let p be a prime number. We recall the notion of *global F-regularity* for normal varieties over an F-finite field of characteristic p, and review the local constraints it imposes, as well as the global constraints it imposes on proper varieties.

Definition 3.8 ([Smi00; SS10])**.** A normal F-finite scheme X over $\mathbb{F}_p$ is called *globally F-regular* if for any effective Weil divisor D on X there exists a positive integer $e \in \mathbb{Z}_{>0}$ such that the map

$$\mathcal{O}_X \to F^e_*\mathcal{O}_X \xrightarrow{F^e_*(\sigma_D)} F^e_*\mathcal{O}_X(D)$$

of $\mathcal{O}_X$-modules splits. Here $\sigma_D\colon \mathcal{O}_X \to \mathcal{O}_X(D)$ is the morphism determined by the Weil divisor D.
A normal F-finite scheme X over $\mathbb{F}_p$ is called *F-split* if $\mathcal{O}_X \to F_*\mathcal{O}_X$ splits.
A pair (X, Δ) consisting of a normal F-finite scheme X over $\mathbb{F}_p$ and an effective $\mathbb{Q}$-Weil divisor Δ is called *globally F-regular* if for any effective Weil divisor D on X there exists an integer $e > 0$ such that the natural map

$$\mathcal{O}_X \to F^e_*\mathcal{O}_X(\lceil (p^e - 1)\Delta\rceil + D)$$

of $\mathcal{O}_X$-modules splits. In particular, X is globally F-regular if and only if $(X, 0)$ is globally F-regular.

The following well-known proposition gives local constraints on a normal variety over an F-finite field to be globally F-regular and echoes Proposition 3.5.

Proposition 3.9. *Let X be a normal F-finite scheme over $\mathbb{F}_p$. If X is globally F-regular, then its local rings are strongly F-regular.*

Proof. We follow the arguments of [Smi19, Prop. 6.22]. Fix a point $x \in X$. If $c \in \mathcal{O}_{X,x}$ is a nonzero element, c defines an effective divisor in the neighborhood of x. By taking the Zariski closure, this divisor extends to a Weil divisor D on X such that the map $\mathcal{O}_X \to \mathcal{O}_X(D)$ localizes to the map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}[c^{-1}]$ sending $1 \mapsto 1$. Since X is globally F-regular, there exists $e > 0$ such that $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ splits. Thus, localizing yields a splitting of the map $\mathcal{O}_{X,x} \to F^e_*(\mathcal{O}_{X,x}[c^{-1}])$ sending $1 \mapsto 1$. Multiplying by c yields a splitting of the map $\mathcal{O}_{X,x} \to F^e_*\mathcal{O}_{X,x}$ sending $1 \mapsto F^e_*c$. □

Global F-regularity is a local property for normal affine varieties. Indeed, a normal affine variety over an F-finite field k is globally F-regular if and only if it is strongly F-regular if and only if all its local rings are strongly F-regular ; see [HH89, Thm. 3.1]. By Proposition 3.5 and the discussion that follows, we see that a $\mathbb{Q}$-Gorenstein normal affine variety over an F-finite field is globally F-regular if and only if it is a splinter. In fact, any normal globally F-regular variety over an F-finite field is a splinter :

Proposition 3.10 ([Bha12, Prop. 8.9], [Bha+22, Lem. 6.14]). *Let X be an F-finite normal excellent scheme over $\mathbb{F}_p$ and let Δ be an effective $\mathbb{Q}$-Weil divisor. If (X, Δ) is globally F-regular, then (X, Δ) is globally $+$-regular.*
In particular, by Remark 3.4, assuming X is a normal scheme of finite type over an F-finite field, if X is globally F-regular, then X is a splinter.

As Proposition 3.6 in the splinter case, global F-regularity imposes strong constraints on the global geometry of normal *projective* varieties :

Proposition 3.11 ([Smi00, Cor. 4.3], [SS10, Thm. 1.1]). *Let X be a normal projective variety over an F-finite field of positive characteristic. Assume that X is globally F-regular. Then :*

(i) *For all nef line bundles $\mathcal{L}$ on X, $H^i(X, \mathcal{L}) = 0$ for all $i > 0$.*
(ii) *X is log Fano ; in particular, if X is in addition $\mathbb{Q}$-Gorenstein, then $-K_X$ is big.*

A proper normal curve over an algebraically closed field k of positive characteristic is globally F-regular if and only if it is splinter if and only if it is isomorphic to the projective line. It follows from the proof of [KT23, Thm. 5.2] that a smooth projective Fano variety over a perfect field of positive characteristic is globally F-regular if and only if it is a splinter if and only if it is F-split. The following conjecture stems from Proposition 3.11(ii) and the folklore expectation (see, e.g., [Bha+22, Rmk. 6.16]) that splinters should be globally F-regular.

Conjecture 3.12 ([Bha+22, Conj. 6.17]). *Let X be a $\mathbb{Q}$-Gorenstein projective scheme over a field of positive characteristic. If X is a splinter, then $-K_X$ is big.*

4. Invariance under small birational maps

We start by describing the behavior of the splinter property under open embeddings. Recall that the normalization of a Nagata scheme is finite [Sta23, Tag 035S], and that, if $Y \to X$ is locally of finite type and X is Nagata, then Y is Nagata [Sta23, Tag 0359]. Moreover, any excellent scheme is Nagata [Sta23, Tag 07QS].

Lemma 4.1. *Let X be a Noetherian scheme and let $U \subseteq X$ be an open dense subset.*

(i) *If X is a splinter, then U is a splinter.*
(ii) *Assume that X is normal and Nagata, and that $\operatorname{codim}_X(X \setminus U) \geq 2$. If U is a splinter, then X is a splinter.*

Proof. To prove (i) consider a finite cover $f \colon Y \to U$. By [Bha12, Prop. 4.1] this cover extends to a finite morphism $\bar{f} \colon \overline{Y} \to X$. Since X is a splinter, we obtain a section s such that the composition

$$\mathcal{O}_X \to \bar{f}_* \mathcal{O}_{\overline{Y}} \xrightarrow{s} \mathcal{O}_X$$

is the identity. By restricting to U, we obtain the desired section of $\mathcal{O}_U \to f_* \mathcal{O}_Y$.

For (ii) consider a finite cover $f \colon Y \to X$. Since X is Nagata, so is Y. By possibly replacing Y by its normalization, we can assume that Y is normal. The sheaf $f_* \mathcal{O}_Y$ satisfies the property S_2 by [Gro65, Prop. 5.7.9] and is therefore reflexive. Since U is a splinter, we obtain a splitting of $\mathcal{O}_U \to f_* \mathcal{O}_Y|_U$ and this extends to a splitting of $\mathcal{O}_X \to f_* \mathcal{O}_Y$ as X is normal and all the involved sheaves are reflexive. □

Remark 4.2. Let Δ be an effective $\mathbb{Q}$-Weil divisor on a normal excellent Noetherian scheme X and let $U \subseteq X$ be an open dense subset. As the proof is analogous to the one of Lemma 4.1, we leave it to the reader to verify that if (X, Δ) is globally $+$-regular, then $(U, \Delta|_U)$ is globally $+$-regular. Conversely, if $\operatorname{codim}_X(X \setminus U) \geq 2$ and if $(U, \Delta|_U)$ is globally $+$-regular, then (X, Δ) is globally $+$-regular.

Lemma 4.3. *Let $\pi \colon X \to Y$ be a surjective morphism of Noetherian schemes. Assume that X is a splinter and that Y is integral with generic point η, then the generic fiber X_η is a splinter.*

Proof. Let $f \colon Z \to X_\eta$ be a finite surjective morphism. Then there exists a nonempty open subset $U \subseteq Y$ such that f spreads out to a finite surjective morphism $f_U \colon Z_U \to X_U := \pi^{-1}(U)$. By Lemma 4.1 the scheme X_U is a splinter. Thus $\mathcal{O}_{X_U} \to f_{U*} \mathcal{O}_{Z_U}$ admits a section. By flat base change, restricting to the generic fiber yields the desired splitting. □

We now turn to the analogues of Lemma 4.1 and Lemma 4.3 in the global F-regular setting. The following elementary lemma appears, e.g., in [Gon+15, Lem. 1.5].

Lemma 4.4. *Let (X, Δ) be a pair consisting of a normal variety over an F-finite field k of positive characteristic and of an effective $\mathbb{Q}$-Weil divisor Δ, and let $U \subseteq X$ be an open subset.*

(i) *If (X, Δ) is globally F-regular, then $(U, \Delta|_U)$ is globally F-regular.*
(ii) *Assume that $U \subseteq X$ is dense and $\operatorname{codim}_X(X \setminus U) \geq 2$. If $(U, \Delta|_U)$ is globally F-regular, then (X, Δ) is globally F-regular.*

Proof. For the proof of (i), consider an effective Weil divisor D_0 on U. Let D be the Zariski closure of D_0 in X. By assumption there exists $e > 0$ such that $\mathcal{O}_X \to F^e_*\mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$ splits. Restricting to U yields the desired splitting of $\mathcal{O}_U \to F^e_*\mathcal{O}_U(\lceil (p^e - 1)\Delta|_U \rceil + D_0)$.

To prove (ii), note that, since F is finite, F_* preserves S_2 sheaves by [Gro65, Prop. 5.7.9], so F_* sends reflexive sheaves to reflexive sheaves. Furthermore, the restriction along $U \hookrightarrow X$ induces a bijection between Weil divisors of X and Weil divisors of U. Statement (ii) follows since, for any Weil divisor D, the map $\mathcal{O}_X \to F^e_*\mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$ of reflexive sheaves splits if and only if its restriction to U splits. $\square$

Lemma 4.5. *Let $\pi \colon X \to Y$ be a surjective morphism of normal varieties over an F-finite field of positive characteristic. Assume that (X, Δ) is globally F-regular and that Y is integral with generic point η. Then the generic fiber X_η is normal and (X_η, Δ_η) is globally F-regular.*

Proof. By [SS10, Lem. 3.5], X is globally F-regular. It follows from Proposition 3.10 that X is a splinter and then from Lemma 4.3 that X_η is a splinter, so X_η is normal. Given an effective Weil divisor D_0 on X_η, we denote by D its Zariski closure in X. Since (X, Δ) is globally F-regular, there exists $e > 0$ such that $\mathcal{O}_X \to F^e_*\mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$ splits. The lemma follows by restricting the splitting along $X_\eta \hookrightarrow X$. $\square$

Finally, we draw as a consequence of the above that both the splinter property and the global F-regular property are invariant under *small birational maps* of normal varieties.

Definition 4.6 (Small birational map)**.** A rational map $f \colon X \dashrightarrow Y$ between Noetherian schemes is said to be a *small birational map* if there exist nonempty open subsets $U \subseteq X$ and $V \subseteq Y$ with $\operatorname{codim}_X(X \setminus U) \geq 2$ and $\operatorname{codim}_Y(Y \setminus V) \geq 2$ such that f induces an isomorphism $U \xrightarrow{\simeq} V$.

Proposition 4.7. *Let $f \colon X \dashrightarrow Y$ be a small birational map between normal schemes of finite type over a Noetherian scheme S. The following statements hold:*

(i) *Assuming S is Nagata, X is a splinter if and only if Y is a splinter.*
(ii) *Assuming $S = \operatorname{Spec} k$ with k of positive characteristic and F-finite, X is globally F-regular if and only if Y is globally F-regular.*

Proof. Statement (i) follows directly from Lemma 4.1 and statement (ii) from Lemma 4.4. $\square$

5. Lifting and descending the splinter property

5.1. **Descending the splinter property.** If $\pi \colon Y \to X$ is a morphism of varieties such that $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ is an isomorphism, e.g., if π is flat proper with geometrically connected and geometrically reduced fibers, or if $\pi \colon Y \to X$ is birational and X is a normal proper variety, it is a formal consequence that the splinter property descends along π. Indeed, we have the following lemma.

Lemma 5.1. *Let $\pi \colon Y \to X$ be a morphism of Noetherian schemes.*

(i) *If Y is a splinter and the map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ is split, then X is a splinter.*
(ii) *If Y is a derived splinter and $\mathcal{O}_X \to \mathrm{R}\pi_*\mathcal{O}_Y$ is split, then X is a derived splinter.*

Assume further that X and Y are normal and excellent, and let Δ be an effective $\mathbb{Q}$-Weil divisor on X.

(iii) *If $(Y, \pi^*\Delta)$ is globally $+$-regular and $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ is split, then (X, Δ) is globally $+$-regular.*

Proof. We only prove (iii) since (i) and (ii) admit similar proofs. Let $f\colon Z \to X$ be a finite surjective morphism with Z normal. Let $Z' \to Y \times_X Z$ be the normalization of the fiber product, which is finite over Y since Y is excellent. (This step is not needed for the proofs of (i) and (ii) where one simply takes $Z' = Y \times_X Z$.) We obtain a commutative square

$$\begin{tikzcd} Z' \arrow[r, "\pi'"] \arrow[d, "f'"] & Z \arrow[d, "f"] \\ Y \arrow[r, "\pi"] & X. \end{tikzcd}$$

Since $(Y, \pi^*\Delta)$ is globally $+$-regular, we have a splitting

$$\mathcal{O}_Y \to f'_*\mathcal{O}_{Z'}(\lfloor f'^*\pi^*\Delta \rfloor) \xrightarrow{t} \mathcal{O}_Y.$$

Fixing a splitting $\mathcal{O}_X \to \pi_*\mathcal{O}_Y \xrightarrow{s} \mathcal{O}_X$, we obtain a factorization

$$\mathrm{id}_{\mathcal{O}_X} \colon \mathcal{O}_X \to \pi_*\mathcal{O}_Y \to \pi_* f'_*\mathcal{O}_{Z'}(\lfloor f'^*\pi^*\Delta \rfloor) = f_*\pi'_*\mathcal{O}_{Z'}(\lfloor \pi'^* f^*\Delta \rfloor) \xrightarrow{\pi_* t} \pi_*\mathcal{O}_Y \xrightarrow{s} \mathcal{O}_X.$$

Since $\mathcal{O}_X \to f_*\pi'_*\mathcal{O}_{Z'}(\lfloor \pi'^* f^*\Delta \rfloor) = f_*\pi'_*\pi'^*\mathcal{O}_Z(\lfloor f^*\Delta \rfloor)$ factors through $\mathcal{O}_X \to f_*\mathcal{O}_Z(\lfloor f^*\Delta \rfloor)$, the lemma follows. ☐

Remark 5.2. Assume X and Y are schemes over a field k. If $X \times_k Y$ is a splinter, then so are X and Y. Indeed, if $\pi\colon X \times_k Y \to X$ denotes the first projection, then $\pi_*\mathcal{O}_{X\times_k Y} = \mathcal{O}_X \otimes_k H^0(Y, \mathcal{O}_Y)$ and any splitting of the k-linear map $k \to H^0(Y, \mathcal{O}_Y), 1 \mapsto 1_Y$ provides a splitting of the natural map $\mathcal{O}_X \to \pi_*\mathcal{O}_{X\times_k Y}$.

5.2. **Lifting the splinter property.** Recall, e.g. from [Har77, Ch. III, Ex. 6.10], that for a finite morphism $\pi\colon Y \to X$ of Noetherian schemes the exceptional inverse image functor is the functor taking quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ to the quasi-coherent $\mathcal{O}_Y$-modules $\pi^!\mathcal{F} \coloneqq \mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_Y, \mathcal{F})$.

Lemma 5.3. *Let $\pi\colon Y \to X$ be a finite surjective morphism of Noetherian schemes such that either $H^0(Y, \mathcal{O}_Y)$ is a field or $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$. Assume that $\pi^!\mathcal{O}_X \cong \mathcal{O}_Y$.*

(i) *If X is a splinter, then Y is a splinter.*
(ii) *Assume that X is defined over $\mathbb{F}_p$. If X is F-split and if the map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ splits, then Y is F-split.*

Proof. First assume that X is a splinter. Let $f\colon Z \to Y$ be a finite surjective morphism. The splitting of $\eta_f\colon \mathcal{O}_Y \to f_*\mathcal{O}_Z$ is equivalent to the surjectivity of

$$\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_Z, \mathcal{O}_Y) \xrightarrow{-\circ\eta_f} \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y).$$

The adjunction $\pi_* \dashv \pi^!$ [Har77, Ch. III, Ex. 6.10(b)] provides the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_Z, \mathcal{O}_Y) \arrow[r, "-\circ\eta_f"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) \arrow[d, "\cong"] \\ \mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_Z, \pi^!\mathcal{O}_X) \arrow[r, "-\circ\eta_f"] \arrow[d, "="] & \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \pi^!\mathcal{O}_X) \arrow[d, "="] \\ \mathrm{Hom}_{\mathcal{O}_X}(\pi_* f_*\mathcal{O}_Z, \mathcal{O}_X) \arrow[r, "-\circ\pi_*\eta_f"] \arrow[d, "-\circ\eta_{\pi\circ f}"] & \mathrm{Hom}_{\mathcal{O}_X}(\pi_*\mathcal{O}_Y, \mathcal{O}_X) \arrow[d, "-\circ\eta_\pi"] \\ \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \arrow[r, equal] & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X). \end{tikzcd}$$

Since X is a splinter, the bottom-left vertical arrow $-\circ\eta_{\pi\circ f}$ is surjective. This implies on the one hand that $-\circ\eta_f$ is nonzero, and on the other hand that $-\circ\eta_\pi$ is surjective. Assuming that $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) = H^0(Y, \mathcal{O}_Y)$ is a field, the former yields that $-\circ\eta_f$ is surjective since it is a map of $H^0(Y, \mathcal{O}_Y)$-modules. Assuming that $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$, the latter yields that the composition of the right vertical arrows, which is $H^0(X, \mathcal{O}_X)$-linear, is bijective and hence that $-\circ\eta_f$ is surjective.

In case where X is assumed to be F-split, we argue via the same diagram with $f\colon Z \to Y$ replaced by the Frobenius $F\colon Y \to Y$. If $s\colon \pi_*\mathcal{O}_Y \to \mathcal{O}_X$ is a splitting of the map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$, then, with a Frobenius splitting of X, we obtain a diagram

$$\mathcal{O}_X \to \pi_* F_* \mathcal{O}_Y = F_*\pi_*\mathcal{O}_Y \xrightarrow{F_*(s)} F_*\mathcal{O}_X \to \mathcal{O}_X,$$

where the composition is the identity. This proves that the bottom-left arrow $-\circ\eta_{\pi\circ F}$ is surjective. As in the splinter case, we deduce that $-\circ\eta_F$ is surjective, i.e., that Y is F-split. □

A morphism of schemes $\pi\colon Y \to X$ is said to be a *crepant morphism* if it is proper, birational, and is such that $\pi^!\mathcal{O}_X \cong \mathcal{O}_Y$. Note that by Lemma 2.2, if X is assumed to be Gorenstein, the latter condition is equivalent to $\pi^*\omega_X = \omega_Y$. We have the following derived version of Lemma 5.3(i) :

Proposition 5.4 (The splinter property in positive characteristic lifts along crepant morphisms)**.** *Let $\pi\colon Y \to X$ be a proper surjective morphism of Noetherian schemes such that either $H^0(Y,\mathcal{O}_Y)$ is a field or $H^0(X,\mathcal{O}_X) = H^0(Y,\mathcal{O}_Y)$. Assume that $\pi^!\mathcal{O}_X \cong \mathcal{O}_Y$. If X is a derived splinter, then Y is a derived splinter.*

In particular :

(i) *If $\pi\colon Y \to X$ is a crepant morphism of Noetherian schemes and if X is a derived splinter, then Y is a derived splinter.*
(ii) *If $\pi\colon Y \to X$ is a crepant morphism of Noetherian schemes over $\mathbb{F}_p$ and if X is a splinter, then Y is a splinter.*

Proof. The statement about derived splinters is proved as Lemma 5.3(i) by using the formalism of the exceptional inverse image functor as described in [Sta23, Tag 0A9Y] and the adjunction $\mathrm{R}\pi_* \dashv \pi^!$. The statement about splinters follows from the fact [Bha12, Thm. 1.4] that splinters in positive characteristic agree with derived splinters. □

Remark 5.5. Statements (i) and (ii) of Proposition 5.4 can also be obtained as a consequence of Theorem 11.15 below.

Remark 5.6. Under the additional assumption that X is Gorenstein, Brion and Kumar establish in [BK05, Lem. 1.3.13] that if $\pi\colon Y \to X$ is a crepant morphism of normal F-finite schemes over $\mathbb{F}_p$ with X F-split, then Y is F-split.

Remark 5.7. Let $\pi\colon Y \to X$ be a birational morphism of normal integral schemes proper over a complete Noetherian local domain with positive characteristic residue field. Assume that X is $\mathbb{Q}$-Gorenstein and that $\pi^*K_X = K_Y$ in $\mathrm{Pic}(Y)\otimes\mathbb{Q}$. It is shown in [Bha+22, Lem. 4.19] that if X is a splinter, then Y is a splinter.

For the sake of completeness, we mention that Lemma 5.3 also holds for globally $+$-regular pairs :

Lemma 5.8. *Let $\pi\colon Y \to X$ be a finite surjective morphism of normal excellent Noetherian schemes such that either $H^0(Y,\mathcal{O}_Y)$ is a field or $H^0(X,\mathcal{O}_X) = H^0(Y,\mathcal{O}_Y)$, and let Δ be an effective $\mathbb{Q}$-Weil divisor on X. Assume that $\pi^!\mathcal{O}_X \cong \mathcal{O}_Y$. If (X,Δ) is globally $+$-regular, then $(Y,\pi^*\Delta)$ is globally $+$-regular.*

Proof. Let $f\colon Z \to Y$ be a finite surjective morphism with Z normal. We have to show that the map $\eta_{f,\pi^*\Delta}\colon \mathcal{O}_Y \to f_*\mathcal{O}_Z(\lfloor f^*\pi^*\Delta\rfloor)$ splits, or equivalently, that

$$\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_Z(\lfloor f^*\pi^*\Delta\rfloor),\mathcal{O}_Y) \xrightarrow{-\circ\eta_{f,\pi^*\Delta}} \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y,\mathcal{O}_Y)$$

is surjective. Note that the map $\eta_{\pi\circ f,\Delta}\colon \mathcal{O}_X \to \pi_* f_* \mathcal{O}_Z(\lfloor f^*\pi^*\Delta\rfloor)$ factors through $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$. Hence, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_Z(\lfloor f^*\pi^*\Delta\rfloor), \mathcal{O}_Y) \arrow[r, "-\circ\eta_{f,\pi^*\Delta}"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_Z(\lfloor f^*\pi^*\Delta\rfloor), \pi^!\mathcal{O}_X) \arrow[r, "-\circ\eta_{f,\pi^*\Delta}"] \arrow[d, "="] & \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \pi^!\mathcal{O}_X) \arrow[d, "="] \\
\mathrm{Hom}_{\mathcal{O}_X}(\pi_*f_*\mathcal{O}_Z(\lfloor f^*\pi^*\Delta\rfloor), \mathcal{O}_X) \arrow[r, "-\circ\pi_*\eta_{f,\pi^*\Delta}"] \arrow[d, "-\circ\eta_{\pi\circ f,\Delta}"] & \mathrm{Hom}_{\mathcal{O}_X}(\pi_*\mathcal{O}_Y, \mathcal{O}_X) \arrow[d, "-\circ\eta_\pi"] \\
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \arrow[r, equal] & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X).
\end{tikzcd}$$

We conclude as in Lemma 5.3. ☐

5.3. **The splinter property and base change of field.** The following proposition establishes the invariance fo the splinter property for proper schemes over a field under algebraic base change. We refer to Remark 6.10 for base change along any field extension.

Proposition 5.9. *Let X be a scheme over a field k such that $H^0(X,\mathcal{O}_X)$ is a field, e.g., X is proper, connected, and reduced. Then, for any algebraic field extension L of $H^0(X,\mathcal{O}_X)$, the scheme $X_L := X\times_{H^0(X,\mathcal{O}_X)} L$ is a splinter if and only if X is a splinter.*

Proof. The "only if" part of the proposition follows from Lemma 5.1. Let $K := H^0(X,\mathcal{O}_X)$. First assume L is a finite extension of K and let $h\colon \operatorname{Spec} L \to \operatorname{Spec} K$. Then

$$h^!\mathcal{O}_{\operatorname{Spec} K} = \mathrm{Hom}_K(L,K) \cong L = \mathcal{O}_{\operatorname{Spec} L}$$

as L-vector spaces. By base change for the exceptional inverse image [Sta23, Tag 0E9U], there exists an isomorphism $\pi^!\mathcal{O}_X \cong \mathcal{O}_{X_L}$. We conclude by Lemma 5.3 that X_L is a splinter, since

$$H^0(X_L,\mathcal{O}_{X_L}) = H^0(X,\mathcal{O}_X)\times_K L = L$$

by flat base change. Now assume $K\to L$ is any algebraic field extension and take a finite cover $f\colon Y\to X_L$. Since f is defined by finitely many equations and L is algebraic over K, we can find a finite cover $f'\colon Y'\to X_{L'}$ such that $K\subseteq L'\subseteq L$ is an intermediate extension, finite over K, and $f = f'\times_{L'} L$ is the base change of f'. By the previous argument $X_{L'}$ is a splinter, thus $\mathcal{O}_{X_{L'}} \to f'_*\mathcal{O}_{Y'}$ admits a section $s\colon f'_*\mathcal{O}_{Y'}\to \mathcal{O}_{X_{L'}}$. Now pulling back to X_L and using flat base change, we obtain the desired section of $\mathcal{O}_{X_L}\to f_*\mathcal{O}_Y$. ☐

Corollary 5.10. *Let X be a connected proper scheme over a field k such that $H^0(X,\mathcal{O}_X)$ is a separable extension of k. Then, for any algebraic field extension K of k, the scheme $X_K := X\times_k K$ is a splinter if and only if X is a splinter.*

Proof. The "only if" part of the corollary follows from Lemma 5.1. Let $\bar{k}$ be an algebraic closure of k. By the assumption on $H^0(X,\mathcal{O}_X)$, $X_{\bar{k}} := X\times_k \bar{k}$ is isomorphic to the disjoint union of $\dim_k H^0(X,\mathcal{O}_X)$ copies of $X\times_{H^0(X,\mathcal{O}_X)} \bar{k}$. By the "if" part of Proposition 5.9, if X is a splinter, then $X_{\bar{k}}$ is a splinter. By Lemma 5.1, we get that X_K is a splinter. ☐

Corollary 5.11. *Let X be a connected proper scheme over a field k. If X is a splinter, then $H^0(X,\mathcal{O}_X)$ is a field, and X, considered as a scheme over* $\operatorname{Spec} H^0(X,\mathcal{O}_X)$*, is geometrically normal.*

Proof. By Proposition 3.5, a splinter is normal. In particular, $H^0(X,\mathcal{O}_X)$ is a field. The corollary then follows from Proposition 5.9. ☐

Remark 5.12 (Algebraic base change for globally $+$-regular pairs). Let X be a connected normal proper scheme over a field k such that $H^0(X,\mathcal{O}_X)$ is a field, and let Δ be an effective $\mathbb{Q}$-Weil divisor on X. Let L be an algebraic field extension of $H^0(X,\mathcal{O}_X)$. The arguments of the proof of Proposition 5.9 show that if (X,Δ) is globally $+$-regular, then $X_L := X\times_{H^0(X,\mathcal{O}_X)} L$ is normal and the pair (X_L,Δ_L) is globally $+$-regular.

Assume in addition that $H^0(X, \mathcal{O}_X)$ is a separable extension of k, and let K be any algebraic extension of k. As in Corollary 5.10, we have that if (X, Δ) is globally $+$-regular, then $X_K \coloneqq X \times_k K$ is normal and the pair (X_K, Δ_K) is globally $+$-regular.

6. Lifting and descending global F-regularity

The first aim of this section is to show that the results of Section 5 regarding splinters, notably Lemma 5.3, extend to the globally F-regular setting; see Proposition 6.4. The second aim is to show how the criterion of Schwede–Smith (recalled in Theorem 6.1) can be used to establish further results in the global F-regular setting; for instance, we show in Proposition 6.7 that global F-regularity is stable under product and, combined with Proposition 6.4, use it to recover in Proposition 6.9 a result of [Gon+15] stating that global F-regularity for normal proper schemes over an F-finite field is stable under base change of fields. Except for Theorem 6.1, which is due to Schwede–Smith, the results of this section will not be used in the rest of the paper.

6.1. **A criterion for global F-regularity.** The following criterion of Schwede–Smith makes it possible in practice to reduce checking that a variety is globally F-regular to simply check that $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ splits for one specific Weil divisor D.

Theorem 6.1 ([SS10, Thm. 3.9])**.** *Let X be a normal variety over an F-finite field of positive characteristic. Then X is globally F-regular if and only if there exists an effective Weil divisor D on X such that*

(i) *there exists an $e > 0$ such that the natural map $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ splits, and*
(ii) *the variety $X \setminus D$ is globally F-regular.*

Remark 6.2. Suppose X is a normal projective variety over an F-finite field of positive characteristic. If D is an ample divisor on X, the variety $X \setminus D$ is affine and therefore globally F-regular if and only if its local rings are strongly F-regular. Since regular local rings are strongly F-regular, a smooth projective variety X over an F-finite field of positive characteristic is globally F-regular if and only if $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ splits for some ample divisor D.

6.2. **Descending global F-regularity.** The following Lemma 6.3, which is due to Schwede–Smith, holds in particular when the map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ is an isomorphism, e.g., when $\pi\colon Y \to X$ is flat proper with geometrically connected and geometrically reduced fibers, or when $\pi\colon Y \to X$ is birational and X is a normal proper variety.

Lemma 6.3 ([SS10, Cor. 6.4])**.** *Let $\pi\colon Y \to X$ be a morphism of varieties over an F-finite field of positive characteristic. If Y is normal globally F-regular and if the map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ is split, then X is normal globally F-regular.*

Proof. By Proposition 3.10, Y is a splinter and, by Lemma 5.1, X is a splinter. Hence, by Proposition 3.5, X is normal. We can now apply [SS10, Cor. 6.4]. □

6.3. **Lifting global F-regularity.** The following Proposition 6.4 is the analogue of Lemma 5.3 and Proposition 5.4. Under the more restrictive assumptions that X is projective and Gorenstein and that π is birational, Proposition 6.4(ii) was previously established in [GT16, Lem. 3.3].

Proposition 6.4. *Let $\pi\colon Y \to X$ be a proper surjective morphism of separated schemes of finite type over an F-finite field k of characteristic $p > 0$ such that either $H^0(Y, \mathcal{O}_Y)$ is a field or $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$. Assume that $\pi^!\mathcal{O}_X \cong \mathcal{O}_Y$, and assume either of the following conditions:*

(i) *π is finite, or*
(ii) *X is quasi-projective.*

If X is a normal globally F-regular variety, then Y is a normal globally F-regular variety.

Proof. We first prove (i). By Proposition 3.10, X is a splinter, and it follows from Lemma 5.3 that Y is a splinter and hence is normal. By normality of X, the complement of X_{reg} has codimension at least 2, and by finiteness of π, the complement of $\pi^{-1}(X_{\mathrm{reg}})$ in Y has codimension at least 2. Therefore, by Lemma 4.4, Y is globally F-regular if and only if $\pi^{-1}(X_{\mathrm{reg}}) \subseteq Y$ is globally F-regular. Replacing X by X_{reg} we can and do assume that X is regular. Since Y is normal, Y_{sing} is a proper closed subset of Y and since π is finite, $\pi(Y_{\mathrm{sing}}) \subseteq X$ is also a proper closed subset. Let $U \subseteq X$ be an affine open subset in the complement of $\pi(Y_{\mathrm{sing}})$. By [Sta23, Tag 0BCU], $D \coloneqq X \setminus U$ has codimension 1, so defines a Weil divisor on X. Since X is regular, D is further Cartier and $\pi^*\mathcal{O}_X(D)$ is a line bundle. Note that the pullback $\pi^*D = \pi^{-1}(D)$ of D is a Cartier divisor [Sta23, Tag 02OO]. Let $\sigma_D \colon \mathcal{O}_X \to \mathcal{O}_X(D)$ be the global section defined by the divisor D. Then the pullback $\pi^*\sigma_D$ defines a global section of $\pi^*\mathcal{O}_X(D)$ whose zero locus is precisely $\pi^{-1}(D)$. Thus $\pi^*\sigma_D$ is the global section of $\pi^*\mathcal{O}_X(D) = \mathcal{O}_Y(\pi^*D)$ defined by the divisor π^*D [Sta23, Tag 0C4S].

Now $\pi^{-1}(U) = Y \setminus \pi^*D$ is an affine open subset contained in the regular locus of Y, so is strongly F-regular. By Theorem 6.1, it is enough to show that there exists an $e > 0$ such that the map $\mathcal{O}_Y \to F^e_*\mathcal{O}_Y(\pi^*D)$ splits. Since X is globally F-regular, there exists an integer $e > 0$ and a splitting s such that

$$\mathrm{id}_{\mathcal{O}_X} \colon \mathcal{O}_X \to F^e_*\mathcal{O}_X \xrightarrow{F^e_*(\sigma_D)} F^e_*\mathcal{O}_X(D) \xrightarrow{s} \mathcal{O}_X.$$

Since X is a splinter, we have a splitting

$$\mathrm{id}_{\mathcal{O}_X} \colon \mathcal{O}_X \xrightarrow{\eta} \pi_*\mathcal{O}_Y \xrightarrow{t} \mathcal{O}_X.$$

Here $\eta \colon \mathrm{id}_{\mathrm{Coh}\,X} \to \pi_*\pi^*$ is the counit of the adjunction $\pi^* \dashv \pi_*$. Note that by the projection formula, t induces a splitting of $\eta \colon \mathcal{O}_X(D) \to \pi_*\pi^*\mathcal{O}_X(D)$ which by abuse we still denote by t. The commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X \arrow[rr, "\eta"] \arrow[d, "f^e"] & & \pi_*\pi^*\mathcal{O}_X \arrow[r, equal] \arrow[d, "f^e"] & \pi_*\mathcal{O}_Y \arrow[d, "f^e"] \\
F^e_*\mathcal{O}_X \arrow[rr, "F^e_*(\eta)"] \arrow[d, "F^e_*(\sigma_D)"] & & F^e_*\pi_*\pi^*\mathcal{O}_X \arrow[r, equal] \arrow[d, "F^e_*\pi_*\pi^*(\sigma_D)"] & \pi_*F^e_*\mathcal{O}_Y \arrow[d, "\pi_*F^e_*(\pi^*\sigma_D)"] \\
F^e_*\mathcal{O}_X(D) \arrow[rr, "F^e_*(\eta)"] \arrow[uu, bend left, "s"] & & F^e_*\pi_*\pi^*\mathcal{O}_X(D) \arrow[r, equal] \arrow[ll, bend left, "F^e_*(t)"] & \pi_*F^e_*\mathcal{O}_Y(\pi^*D)
\end{tikzcd}$$

shows that the map

$$\pi_*F^e_*(\pi^*\sigma_D) \circ f^e \circ \eta \colon \mathcal{O}_X \to \pi_*F^e_*\mathcal{O}_Y(\pi^*D) \tag{6.5}$$

splits. Here, $f^e \colon \mathcal{O}_X \to F^e_*\mathcal{O}_X$ denotes the p^e-th power map on local sections.

As in Lemma 5.3 we consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_Y}(F^e_*\mathcal{O}_Y(\pi^*D), \mathcal{O}_Y) \arrow[r, "-\circ\pi^*\sigma_D"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{O}_Y}(F^e_*\mathcal{O}_Y(\pi^*D), \pi^!\mathcal{O}_X) \arrow[r, "-\circ\pi^*\sigma_D"] \arrow[d, "="] & \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \pi^!\mathcal{O}_X) \arrow[d, "="] \\
\mathrm{Hom}_{\mathcal{O}_X}(\pi_*F^e_*\mathcal{O}_Y(\pi^*D), \mathcal{O}_X) \arrow[r, "-\circ\pi_*F^e_*(\pi^*\sigma_D)\circ f^e"] \arrow[d, "-\circ\pi_*F^e_*(\pi^*\sigma_D)\circ f^e\circ\eta"] & \mathrm{Hom}_{\mathcal{O}_X}(\pi_*\mathcal{O}_Y, \mathcal{O}_X) \arrow[d, "-\circ\eta_\pi"] \\
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \arrow[r, equal] & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X).
\end{tikzcd}$$

Since the splitting of (6.5) is equivalent to the left-vertical map $- \circ \pi_*F^e_*(\pi^*\sigma_D) \circ f^e \circ \eta$ being surjective, we conclude, as in the proof of Lemma 5.3, that the map $- \circ \pi^*\sigma_D$ is surjective, or equivalently, that $\mathcal{O}_Y \to F^e_*\mathcal{O}_Y(\pi^*D)$ splits.

We now prove (ii). By Proposition 3.10, X is a splinter, so a derived splinters since k is of positive characteristic [Bha12, Thm. 1.4]. It follows from Proposition 5.4 that Y is a splinter and hence is normal. By Remark 2.4, π is generically finite. Let $U \subseteq X$ be an affine regular dense open subset such that the restriction of π to U is finite. After possibly further shrinking U we can assume that $\pi^{-1}(U)$ is also regular. Since restriction to open subschemes commutes with exceptional inverse image functors [Sta23, Tag 0G4J], we have $\pi|_U^!\mathcal{O}_U \cong \mathcal{O}_{\pi^{-1}(U)}$. Since X is quasi-projective, any

Weil divisor on X is dominated by a Cartier divisor. Thus, by possibly further shrinking U, we may and do assume that $D := X \setminus U$ is a Cartier divisor. Since $\pi^{-1}(U)$ is affine an regular, it is enough to show, by Theorem 6.1, that there exists an $e > 0$ such that the map $\mathcal{O}_Y \to F^e_* \mathcal{O}_Y(\pi^* D)$ splits. By using the same diagrams as in (i) with π_* replaced by $\mathrm{R}\pi_*$, we conclude that a splitting of $\mathcal{O}_X \to F^e_* \mathcal{O}_X(D)$ induces a splitting of $\mathcal{O}_Y \to F^e_* \mathcal{O}_Y(\pi^* D)$. □

Remark 6.6. By using the version of Theorem 6.1 for pairs, i.e., the original [SS10, Thm. 3.9], we leave it to the reader to show the following version of Proposition 6.4(i) for pairs. Let $\pi \colon Y \to X$ be a finite surjective morphism of separated schemes of finite type over an F-finite field k of characteristic $p > 0$, and let Δ be an effective $\mathbb{Q}$-Weil divisor on X. Assume that $\pi^! \mathcal{O}_X \cong \mathcal{O}_Y$, and that either $H^0(Y, \mathcal{O}_Y)$ is a field or $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$. If X is normal and if (X, Δ) is globally F-regular, then Y is normal and $(Y, \pi^* \Delta)$ is globally F-regular.

6.4. **Products of globally F-regular varieties.** As far as we know, it is unknown whether the splinter property is stable under product. On the other hand, global F-regularity for products is more tractable since the Frobenius of a product is the product of the Frobenii and since one may use the criterion of Theorem 6.1 to check global F-regularity for one specific divisor. The following Proposition 6.7 generalizes [Has03, Thm. 5.2], where the case of products of projective globally F-regular varieties was dealt with by taking affine cones.

Proposition 6.7. *Let X and Y be normal varieties over a perfect field k of positive characteristic. Then, X and Y are globally F-regular if and only if their product $X \times_k Y$ is globally F-regular.*

Proof. Denote by π_X and π_Y the natural projections from $X \times_k Y$ to X and Y, resp. Since X and Y are normal and k is perfect, their product $X \times_k Y$ is normal, see [Sta23, Tag 038L]. Moreover, $X \setminus X_{\mathrm{reg}}$ and $Y \setminus Y_{\mathrm{reg}}$ both have codimension ≥ 2 and thus $X \times_k Y \setminus X_{\mathrm{reg}} \times_k Y_{\mathrm{reg}}$ has codimension ≥ 2. Therefore, by Lemma 4.4, we can assume without loss of generality that X and Y are smooth over k.

Assume first that $X \times_k Y$ is globally F-regular. As in Remark 5.2, since $\pi_* \mathcal{O}_{X \times_k Y} = \mathcal{O}_X \otimes_k H^0(Y, \mathcal{O}_Y)$, any splitting of the k-linear map $k \to H^0(Y, \mathcal{O}_Y), 1 \mapsto 1_Y$ provides a splitting to the natural map $\mathcal{O}_X \to \pi_* \mathcal{O}_{X \times_k Y}$. By Lemma 6.3, it follows that X is globally F-regular.

For the converse, we first note that there exist effective Cartier divisors D on X and E on Y such that $X \setminus D$ and $Y \setminus E$ are affine. Indeed, since X and Y are normal and k is perfect, they admit dense affine open subsets, and then use the fact that the complement of a dense affine open subset is a divisor by [Sta23, Tag 0BCU]. The divisors obtained this way are a priori Weil divisors, but since X and Y are smooth, they are actually Cartier divisors. Since k is assumed to be perfect,

$$X \setminus D \times_k Y \setminus E = X \times_k Y \setminus \pi_X^* D \cup \pi_Y^* E$$

is smooth and affine, so in particular strongly F-regular. By Theorem 6.1 it is enough to show that the map

$$\mathcal{O}_{X \times Y} \to F^e_* \mathcal{O}_{X \times Y} \to F^e_* \mathcal{O}_{X \times Y}(\pi_X^* D + \pi_Y^* E)$$

splits for some $e > 0$. Since X is globally F-regular, we can find an $e > 0$ such that $\mathcal{O}_X \to F^e_* \mathcal{O}_X(D)$ splits and similarly for Y. As remarked in [Smi00, p. 558], if $\mathcal{O}_X \to F^e_* \mathcal{O}_X(D)$ splits for some $e > 0$, then $\mathcal{O}_X \to F^{e'}_* \mathcal{O}_X(D)$ splits for all $e' \geq e$. Thus, there exists an integer $e > 0$ such that both $\mathcal{O}_X \to F^e_* \mathcal{O}_X(D)$ and $\mathcal{O}_Y \to F^e_* \mathcal{O}_Y(E)$ split. The morphism

$$\sigma_{\pi_X^* D + \pi_Y^* E} \colon \mathcal{O}_{X \times Y} \to \mathcal{O}_{X \times Y}(\pi_X^* D + \pi_Y^* E)$$

can be identified with the tensor product $\pi_X^* \sigma_D \otimes \pi_Y^* \sigma_E$, where

$$\sigma_D \colon \mathcal{O}_X \to \mathcal{O}_X(D) \quad \text{and} \quad \sigma_E \colon \mathcal{O}_Y \to \mathcal{O}_Y(E)$$

denote the corresponding morphisms on X and Y. Pushing forward along Frobenius, we obtain

$$F^e_* \sigma_{\pi_X^* D + \pi_Y^* E} = \pi_X^* F^e_* \sigma_D \otimes \pi_Y^* F^e_* \sigma_E.$$

We conclude, by taking the tensor product of the sections of $\mathcal{O}_X \to F^e_* \mathcal{O}_X(D)$ and $\mathcal{O}_Y \to F^e_* \mathcal{O}_Y(E)$, that

$$\mathcal{O}_{X \times Y} \to F^e_* \mathcal{O}_{X \times Y}(\pi_X^* D + \pi_Y^* E)$$

splits. Hence $X \times_k Y$ is globally F-regular. □

Remark 6.8. By using the version of Theorem 6.1 for pairs, i.e., the original [SS10, Thm. 3.9], we leave it to the reader to show the following version of Proposition 6.7 for pairs: Let X and Y be normal varieties over a perfect field k of positive characteristic, and denote by $\pi_X \colon X \times_k Y \to X$ and $\pi_Y \colon X \times_k Y \to Y$ the natural projections. Let Δ_X and Δ_Y be effective $\mathbb{Q}$-Weil divisors on X and Y, resp. Then, (X, Δ_X) and (Y, Δ_Y) are globally F-regular if and only if $(X \times_k Y, \pi_X^*\Delta_X + \pi_Y^*\Delta_Y)$ is globally F-regular.

6.5. **Global F-regularity and base change of field.** By using the lifting Proposition 6.4, it is possible to show that the base change results for splinters along algebraic field extensions of Section 5.3 also hold for normal global F-regular varieties. However, by using the criterion of Schwede–Smith [SS10, Thm. 3.9], Gongyo–Li–Patakfalvi–Schwede–Tanaka–Zong [Gon+15] have established a more general base change results that deals with not necessarily algebraic extensions.

Proposition 6.9 (Gongyo–Li–Patakfalvi–Schwede–Tanaka–Zong [Gon+15])**.** *Let X be a normal proper scheme over an F-finite field k. Assume that (X, Δ) is globally F-regular. Then, for any F-finite field extension L of k with a morphism $\operatorname{Spec} L \to \operatorname{Spec} H^0(X, \mathcal{O}_X)$, the scheme $X_L := X \times_{H^0(X,\mathcal{O}_X)} L$ is normal and the pair (X_L, Δ_L) is globally F-regular.*

Proof. Let us provide an alternate proof. We may and do assume X is connected. Since any divisor on X_L is defined over a finitely generated field extension of the field $H^0(X, \mathcal{O}_X)$, we may assume that L is a simple extension of $H^0(X, \mathcal{O}_X)$. If L is algebraic, we can apply Proposition 6.4, while if L is purely transcendental, we can apply Proposition 6.7 (or Remark 6.8 in case $\Delta \neq 0$) and Lemma 4.5 to $X \times_{H^0(X,\mathcal{O}_X)} \mathbb{A}^1 \to \mathbb{A}^1$. Note that in this situation it is not necessary to assume that $H^0(X, \mathcal{O}_X)$ is perfect as $X_{\text{reg}} \times_{H^0(X,\mathcal{O}_X)} \mathbb{A}^1$ is regular. □

Remark 6.10. Our proof of Proposition 6.9 shows that one could extend Proposition 5.9 concerned with base change of splinters along algebraic field extensions to arbitrary field extensions if one could establish that the splinter property is stable under taking product with the affine line $\mathbb{A}^1$.

Corollary 6.11 ([Gon+15, Cor. 2.8])**.** *Let X be a connected normal proper scheme over an F-finite field k. Assume that $H^0(X, \mathcal{O}_X)$ is a separable extension of k and that (X, Δ) is globally F-regular. Then, for any F-finite field extension K of k, the scheme $X_K := X \times_k K$ is normal and the pair (X_K, Δ_K) is globally F-regular.*

7. Finite torsors over splinters

We say that a morphism $\pi \colon Y \to X$ of schemes over a scheme S is a *finite torsor* if it is a torsor under a finite group scheme G over S. The aim of this section is to prove Theorem **(A)**. First, in order to apply our lifting Lemma 5.3 to finite torsors over splinters, we have:

Lemma 7.1. *Let $\pi \colon Y \to X$ be a morphism of Noetherian schemes over a Noetherian ring R. Assume that π satisfies either of the following conditions:*

(i) *π is finite étale.*
(ii) *π is a finite torsor, and* $\operatorname{Pic}(\operatorname{Spec} R) = 0$*, e.g. R is a local ring or a UFD.*

Then $\pi^!\mathcal{O}_X \cong \mathcal{O}_Y$.

Proof. Case (i) is covered by [Sta23, Tag 0FWI]. Concerning case (ii), this is claimed in [BM76, p. 222] in the special case where R is a field and we provide here a proof. For finite morphisms, the exceptional inverse image functor is defined at the level of coherent sheaves and we have $\pi_*\pi^!\mathcal{O}_X \cong \mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_Y, \mathcal{O}_X)$; see [Sta23, Tag 0AU3]. Thus to show that $\pi^!\mathcal{O}_X \cong \mathcal{O}_Y$, we must produce an isomorphism of $\pi_*\mathcal{O}_Y$-modules

$$\pi_*\mathcal{O}_Y \xrightarrow{\cong} \mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_Y, \mathcal{O}_X),$$

or equivalently produce an $\mathcal{O}_X$-linear map $\operatorname{Tr}_{Y/X} \colon \pi_*\mathcal{O}_Y \to \mathcal{O}_X$ such that the symmetric bilinear form $\operatorname{Tr}_{Y/X}(\alpha \cdot \beta)$ on the locally free sheaf $\pi_*\mathcal{O}_Y$ with values in $\mathcal{O}_X$ is nonsingular. Such a map is provided for finite G-torsors $Y \to X$ over a field by [CR22, Thm. 3.9]. (Note from e.g. [Bri17, Prop. 2.6.4 & 2.6.5(i)] that any finite G-torsor is a finite G-quotient in the sense of [CR22, Rmk. 2.3].)

In the general case, where R is a Noetherian ring with trivial Picard group, let G be a finite group scheme over R. Since G is flat over R, $H \coloneqq H^0(G, \mathcal{O}_G)$ is a finitely generated projective Hopf algebra with antipode. The dual Hopf algebra $H^\vee = \mathrm{Hom}_R(H, R)$ is also a finitely generated projective Hopf algebra with antipode. Since $\mathrm{Pic}(\operatorname{Spec} R) = 0$, $H^\vee$ admits the additional structure of a Frobenius algebra [Par71, Thm. 1]. By [Par71, Thm. 3 & Discussion on p. 596] the R-submodule $\int^l_{H^\vee} \subseteq H^\vee$ of left integrals is freely generated by a nonsingular left integral $\mathrm{Tr}_G \in H^\vee$. If $\pi\colon Y \to X$ is a G-torsor over R, one constructs by pulling back Tr_G along $X \to \operatorname{Spec} R$ as in [CR22, §3.1, p. 12] an $\mathcal{O}_X$-linear map $\mathrm{Tr}_{Y/X}\colon \pi_*\mathcal{O}_Y \to \mathcal{O}_X$. Since Tr_G is nonsingular, arguing as in [CR22, Thm. 3.9] shows that the bilinear form $(\alpha, \beta) \mapsto \mathrm{Tr}_{Y/X}(\alpha \cdot \beta)$ is nonsingular. □

Remark 7.2. Let $\pi\colon Y \to X$ be a finite torsor with X separated of finite type over a Noetherian ring R such that R admits a dualizing complex and $\mathrm{Pic}(\operatorname{Spec} R) = 0$. As explained in Remark 2.3, Lemma 7.1 implies that $\pi^*\omega_X^\bullet \cong \omega_Y^\bullet$. In particular, if X is Gorenstein, then Y is Gorenstein. Likewise, if X is Cohen–Macaulay, then Y is Cohen–Macaulay.

We have the following lemma from [Bha+22]:

Lemma 7.3 ([Bha+22, Lem. 6.6])**.** *Let $X \to \operatorname{Spec} R$ be a Noetherian Nagata scheme over a ring R. Then the following are equivalent:*

(i) *The scheme X is a splinter.*
(ii) *For each closed point $z \in \operatorname{Spec} R$ the base change to the localization X_{R_z} is a splinter.*

Further, assume X is in addition normal and excellent and let Δ be an effective $\mathbb{Q}$-Weil divisor on X. Then the following are equivalent:

(i) *The pair (X, Δ) is globally $+$-regular.*
(ii) *For each closed point $z \in \operatorname{Spec} R$ the base change to the localization (X_{R_z}, Δ_{R_z}) is globally $+$-regular.*

Proof. Our assumptions are less restrictive than the setup of [Bha+22, §6], but the proof of [Bha+22, Lem. 6.6] works as we outline below. We only consider the case where X is excellent, since the case where X is only Nagata and $\Delta = 0$ is proven by the same argument (note that both conditions imply that X is normal).

Working on each connected component of X separately, we can and do assume that X is integral. If $f\colon Y \to X$ is a finite cover with Y normal, we have to show that the evaluation-at-1 map

$$\mathrm{Hom}_{\mathcal{O}_X}(f_*\mathcal{O}_Y(\lfloor f^*\Delta \rfloor), \mathcal{O}_X) \to H^0(X, \mathcal{O}_X)$$

is surjective. As argued in the proof of [Bha+22, Lem. 6.6], using flat base change this is equivalent to the surjectivity of the evaluation-at-1 map

$$\mathrm{Hom}_{\mathcal{O}_{X_{R_z}}}(f_*\mathcal{O}_Y(\lfloor f^*\Delta|_{X_{R_z}} \rfloor), \mathcal{O}_{X_{R_z}}) \to H^0(X_{R_z}, \mathcal{O}_{X_{R_z}})$$

for every closed point $z \in \operatorname{Spec} R$. To conclude, it is enough to observe that any finite surjective morphism $h\colon Y' \to X_{R_z}$ with normal and integral Y' is the localization of a finite surjective morphism $Y \to X$ with Y normal. Indeed, since X is Nagata, so is X_{R_z} [Sta23, Tag 032U]. Therefore, such $Y \to X$ is provided by taking the normalization of $\mathcal{O}_X$ in the fraction field $K(Y')$, see [Sta23, Tag 0AVK]. □

Proposition 7.4. *Let $\pi\colon Y \to X$ be a morphism of Noetherian Nagata schemes over a Noetherian ring R such that either $H^0(Y, \mathcal{O}_Y)$ is a field or $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$. Assume that π satisfies either of the following conditions:*

(i) *π is finite étale.*
(ii) *π is a finite torsor.*

If X is a splinter, then Y is a splinter.

Proof. Let $z \in \operatorname{Spec} R$ be a closed point. By flat base change $H^0(X_{R_z}, \mathcal{O}_{X_{R_z}}) = H^0(X, \mathcal{O}_X) \otimes_R R_z$ and $H^0(Y_{R_z}, \mathcal{O}_{Y_{R_z}}) = H^0(Y, \mathcal{O}_Y) \otimes_R R_z$. By Lemma 7.3 we can reduce to the case where R is a local ring so that $\mathrm{Pic}(\operatorname{Spec} R) = 0$. From Lemma 7.1 we know that $\pi^!\mathcal{O}_X \cong \mathcal{O}_Y$ for any finite étale or finite torsor morphism $\pi\colon Y \to X$. With the additional assumption that $H^0(Y, \mathcal{O}_Y)$ is a field or $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$, Lemma 5.3 shows that Y is a splinter. □

We say that a morphism of schemes $\pi\colon Y \to X$ over k is *quasi-étale* (resp. a *quasi-torsor*) if there exists $U \subseteq X$ open with $\operatorname{codim}_X(X \setminus U) \geq 2$ such that $f|_{f^{-1}(U)}$ is étale (resp. a torsor under a group scheme G over k). The following proposition, which complements and extends [Bha+22, Prop. 6.20], will not be used in this work but might be of independent interest.

Proposition 7.5. *Let $\pi\colon Y \to X$ be a morphism of normal Noetherian Nagata schemes over a Noetherian ring R such that either $H^0(Y, \mathcal{O}_Y)$ is a field or $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$. Assume that π satisfies either of the following conditions:*

(i) *π is finite quasi-étale.*
(ii) *π is a finite quasi-torsor.*

If X is a splinter, then Y is a splinter.
Assume in addition that X is excellent and let Δ be an effective $\mathbb{Q}$-Weil divisor on X. If (X, Δ) is globally $+$-regular, then $(Y, \pi^*\Delta)$ is globally $+$-regular.

Proof. We only provide a proof in case X is excellent and (X, Δ) is globally $+$-regular since the case where X is only assumed to be Nagata and a splinter follows by the same argument with $\Delta = 0$.

Using Lemma 7.3 we can reduce to the case where R is local so that $\operatorname{Pic}(\operatorname{Spec} R) = 0$. By assumption, there exists an open subset $U \subseteq X$ such that $\operatorname{codim}_X(X \setminus U) \geq 2$ and such that $\pi|_U\colon V := \pi^{-1}(U) \to U$ is finite étale or a finite torsor. Thus, by Lemma 7.1, $\pi_U^! \mathcal{O}_U \cong \mathcal{O}_V$. Since Y is assumed to be normal, we can work on each connected component of Y separately and assume without loss of generality that Y is connected. Since π is finite, $Y \setminus V$ has codimension at least 2 in Y. Since X and Y are normal, we have $H^0(U, \mathcal{O}_U) = H^0(X, \mathcal{O}_X)$ and $H^0(V, \mathcal{O}_V) = H^0(Y, \mathcal{O}_Y)$. Lemma 5.8 shows that $(V, \pi^*\Delta|_V)$ is globally $+$-regular. We conclude with Remark 4.2 that $(Y, \pi^*\Delta)$ is globally $+$-regular. □

Remark 7.6. By replacing the use of Lemma 5.8 with Proposition 6.4 (or rather Remark 6.6) and the use of Remark 4.2 with Lemma 4.4 in the proof of Proposition 7.5, one obtains the following statement. Let $\pi\colon Y \to X$ be a finite morphism of normal varieties over an F-finite field such that either $H^0(Y, \mathcal{O}_Y)$ is a field or $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$. Assume that π is quasi-étale or a quasi-torsor, and let Δ be an effective $\mathbb{Q}$-Weil divisor on X. If (X, Δ) is globally F-regular, then $(Y, \pi^*\Delta)$ is globally F-regular. This extends [PZ20, Lem. 11.1] where the quasi-étale case was treated.

We now focus on finite torsors over proper splinters over a field. The following lemma can be found in [Mum70, Thm. 2, p. 121] (we thank Michel Brion for bringing this reference to our attention). We provide an alternate proof based on Hirzebruch–Riemann–Roch for (not necessarily smooth) proper schemes over a field.

Lemma 7.7. *Let X be a proper scheme over a field k and let $\pi\colon Y \to X$ be a morphism of schemes over k. Assume that π satisfies either of the following conditions:*

(i) *π is finite étale.*
(ii) *π is a finite torsor.*

Then $\chi(\mathcal{O}_Y) = \deg(\pi)\chi(\mathcal{O}_X)$.

Proof. We first establish (ii). Recall that there is a Hirzebruch–Riemann–Roch formula

$$\chi(\mathcal{E}) = \int_X \operatorname{ch}(\mathcal{E}) \cap \operatorname{td}(X)$$

for any vector bundle $\mathcal{E}$ on a proper scheme X over a field; see [Ful98, Cor. 18.3.1]. In particular, the Euler characteristic only depends on the class of the Chern character $\operatorname{ch}(\mathcal{E}) \in \mathrm{A}^*(X)_{\mathbb{Q}}$, where $\mathrm{A}^*(X)_{\mathbb{Q}}$ denotes the Chow cohomology [Sta23, Tag 0FDV] with $\mathbb{Q}$-coefficients. Assume $\pi\colon Y \to X$ is a torsor under a finite group scheme G over k. By definition of a G-torsor, the product $Y \times_X Y \to Y$ is isomorphic to $G \times_k Y \to Y$ as schemes over Y. This gives an isomorphism

$$\pi_*\mathcal{O}_Y \otimes \pi_*\mathcal{O}_Y \cong \pi_*\mathcal{O}_Y^{\oplus n},$$

where $n = \deg(\pi)$ is the order of G. Since π is finite flat, $\pi_*\mathcal{O}_Y$ is a vector bundle (of rank n) on X, and hence by [Sta23, Tag 02UM] we have the identity

$$\mathrm{ch}(\pi_*\mathcal{O}_Y) \cdot \mathrm{ch}(\pi_*\mathcal{O}_Y) = n\ \mathrm{ch}(\pi_*\mathcal{O}_Y) \quad \text{in } \mathrm{A}^*(X)_{\mathbb{Q}}.$$

Since $\mathrm{ch}(\pi_*\mathcal{O}_Y)$ is a unit in $\mathrm{A}^*(X)_{\mathbb{Q}}$, we obtain $\mathrm{ch}(\pi_*\mathcal{O}_Y) = n\,\mathrm{ch}(\mathcal{O}_X)$. By Hirzebruch–Riemann–Roch, the equality $\chi(\pi_*\mathcal{O}_Y) = \chi(\mathcal{O}_Y) = n\chi(\mathcal{O}_X)$ follows. If π is finite étale, one can use Grothendieck–Riemann–Roch for proper schemes over a field [Ful98, Thm. 18.3], while noting that the relative Todd class $\mathrm{td}(T_\pi)$ is equal to 1. Alternatively one can reduce to case (ii) as follows. There exists a Galois cover $\rho\colon Y' \to X$ that dominates π; see [Sta23, Tag 03SF]. We then have a diagram

$$\begin{tikzcd} Y' \arrow[r, "\rho_Y"] \arrow[dr, "\rho_X"'] & Y \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

where ρ_X is a finite $\mathrm{Aut}_X(Y')$-torsor and ρ_Y is a finite $\mathrm{Aut}_Y(Y')$-torsor. □

Theorem 7.8. *Let X be a proper scheme over an integral Noetherian scheme S of positive characteristic and let $\pi\colon Y \to X$ be a morphism of schemes over S. Assume that $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$. In addition, assume either of the following conditions:*

(i) *π is finite étale.*
(ii) *π is a finite torsor.*

If X is a splinter, then π is an isomorphism.

Proof. Let η be the generic point of S. It is is enough to show that the restriction $\pi_\eta\colon Y_\eta \to X_\eta$ of π to η is an isomorphism. By Lemma 4.3, if X is a splinter, then X_η is a splinter. Therefore, we may and do assume that S is the spectrum of a field. Moreover, since a splinter is normal, we may and do assume that X is connected, in which case $H^0(X, \mathcal{O}_X)$ is a field. By Proposition 7.4, Y is a splinter. Since the structure sheaf of a proper splinter in positive characteristic has trivial positive cohomology by Proposition 3.6, we have $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = 1$, where the dimension is taken with respect to the field $H^0(X, \mathcal{O}_X)$ and we conclude with Lemma 7.7 that π is an isomorphism. □

Theorem 7.9. *Let X be a connected proper scheme over a field k of positive characteristic with a k-rational point $x \in X(k)$. Assume that X is a splinter.*

(i) *If k is separably closed, then the étale fundamental group $\pi_1^{\acute{e}t}(X, x)$ of X is trivial.*
(ii) (Theorem **(A)**) *The Nori fundamental group $\pi_1^N(X, x)$ of X is trivial.*

Proof. Statement (i) follows from Theorem 7.8 since for k separably closed any connected finite étale cover $\pi\colon Y \to X$ satisfies $H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X) = k$. For statement (ii), first note that the Nori fundamental group $\pi_1^N(X, x)$ is well-defined as a splinter is reduced. Assume for contradiction that $\pi_1^N(X, x)$ is nontrivial. Since $\pi_1^N(X, x)$ is pro-finite [Nor82], there is a surjective group scheme homomorphism $\pi_1^N(X, x) \twoheadrightarrow G$ to a nontrivial finite group scheme G. By [Nor82, Prop. 3, p. 87], there exists a G-torsor $Y \to X$ with $H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$. This contradicts Theorem 7.8. □

Remark 7.10 (On the triviality of the Nori fundamental group)*.* As mentioned in Corollary 5.11, a connected proper splinter X is geometrically normal, hence geometrically reduced, over the field $K := H^0(X, \mathcal{O}_X)$; in particular it acquires a rational point after some finite separable field extension of K. Moreover, recall the general facts that Nori's fundamental group is invariant under separable base change, and that the triviality of Nori's fundamental group is independent of the choice of base point.

We say that a finite étale cover $\pi\colon Y \to X$ is trivial if it is isomorphic over X to a disjoint union of copies of X. We say that a finite torsor $\pi\colon Y \to X$ under a finite group scheme G over k is trivial if it is isomorphic to $X \times_k G$ over X. An immediate consequence of Theorem 7.9 is the following:

Corollary 7.11. *Let X be a connected proper scheme over a field k of positive characteristic. Assume that X is a splinter.*

(i) *If k is separably closed, then any finite étale cover of X is trivial.*

(ii) *If k is algebraically closed, then any finite torsor over X is trivial.*

Proof. Statement (i) is clear from (the proof of) Theorem 7.9, while statement (ii) follows from the fact [Nor82] that for a k-point of $x \in X(k)$ there is an equivalence of categories between the category of finite torsors $Y \to X$ equipped with a k-point $y \in Y(k)$ mapping to x and the category of finite group schemes G over k equipped with a k-group scheme homomorphism $\pi_1^N(X, x) \to G$. □

8. Proper splinters have negative Kodaira dimension

Let X be a Gorenstein projective scheme over a field k of positive characteristic. Since $-K_X$ big implies that X has negative Kodaira dimension, it is expected in view of Conjecture 3.12 that, if X is a splinter, then its Kodaira dimension is negative. In this section, we confirm this expectation (without assuming X to be Gorenstein) and prove Theorem **(B)**.

Let X be a normal proper variety over a field k and let D be a Weil divisor on X. We define the *Iitaka dimension* of D to be

$$\kappa(X, D) \coloneqq \min\left\{k \mid (h^0(X, \mathcal{O}_X(dD))/d^k)_{d\geq 0} \text{ is bounded}\right\}.$$

By convention, if $h^0(X, \mathcal{O}_X(dD)) = 0$ for all $d > 0$, then we set $\kappa(X, D) = -\infty$. Beware that we deviate from usual conventions as the Iitaka dimension is usually defined for line bundles on projective varieties. If X is a smooth projective variety over k, $\kappa(X, K_X)$ agrees with the Kodaira dimension of X. The following proposition refines the observation from Lemma 3.7 showing that if X is a proper splinter in positive characteristic, then K_X is not effective.

Theorem 8.1 (Theorem **(B)**)**.** *Let X be a positive-dimensional connected proper scheme over a field of positive characteristic. If X is a splinter, then $\kappa(X, K_X) = -\infty$.*

First we have the following variant of a well-known lemma ; see, e.g., [PST17, Ex. 2.12].

Lemma 8.2. *Let X be a proper scheme over a field k of positive characteristic $p > 0$. Assume either of the following conditions :*

(i) *X is a splinter, or*
(ii) *X is normal and F-split, and k is F-finite.*

Then the Weil divisor $(1-p)K_X$ is effective. In particular, either $\kappa(X, K_X) = -\infty$, or K_X is torsion (in which case $\kappa(X, K_X) = 0$).

Proof. First assume that X is a splinter. Since X is normal, we can and do assume that X is connected. After replacing the base field k by $H^0(X, \mathcal{O}_X)$, we consider the base change $\pi\colon X_{\bar{k}} \to X$ along an algebraic closure $k \to \bar{k}$. By Proposition 5.9, $X_{\bar{k}}$ is a splinter. The base change formula for the exceptional inverse image [Sta23, Tag 0E9U] shows that $\pi^*\omega_X = \omega_{X_{\bar{k}}}$. Thus, it is enough to show the statement for $X_{\bar{k}}$, since π flat implies $H^0(X_{\bar{k}}, \pi^*\mathcal{O}_X((1-p)K_X)) = H^0(X, \mathcal{O}_X((1-p)K_X)) \otimes_k \bar{k}$. Since $\bar{k}$ is F-finite, $X_{\bar{k}}$ is in particular F-split. Thus, it is enough to show the statement under the assumptions (ii).

Assume that X is normal, F-split and that k is F-finite. Parts of the arguments below can for example be found in [SS10, §4.2]. We provide nonetheless a proof for the sake of completeness. First note that the absolute Frobenius $F\colon X \to X$ is a finite map. Thus we have for any coherent sheaf $\mathcal{F}$ on X the isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{F}, \omega_X) = F_*\,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, F^!\omega_X).$$

This is clear if X is Cohen–Macaulay as then $\omega_X^\bullet = \omega_X[-\dim X]$, but it is also true if X is only assumed to be normal, by restricting to the Cohen–Macaulay locus and then using that the involved sheaves are reflexive. Choosing a (non-canonical) isomorphism $F^!k = \mathrm{Hom}_k(F_*k, k) \cong F_*k$ as k-vector spaces, we obtain an isomorphism $F^!\omega_X^\bullet \cong \omega_X^\bullet$ and further an isomorphism $F^!\omega_X \cong \omega_X$. In particular, $\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \omega_X) \cong F_*\omega_X$.

By assumption, there exists a map s such that the composition

$$\mathcal{O}_X \to F_*\mathcal{O}_X \xrightarrow{s} \mathcal{O}_X$$

is the identity. We apply $\mathcal{H}om_{\mathcal{O}_X}(-,\omega_X)$ to obtain

$$\omega_X \leftarrow F_*\omega_X \xleftarrow{s^\vee} \omega_X,$$

where the composition is the identity. After restricting to the regular locus, we can twist with $\omega_{X_{\mathrm{reg}}}^{-1}$ and apply the projection formula to obtain a diagram

$$\mathcal{O}_{X_{\mathrm{reg}}} \leftarrow F_*\mathcal{O}_{X_{\mathrm{reg}}}((1-p)K_{X_{\mathrm{reg}}}) \xleftarrow{s^\vee} \mathcal{O}_{X_{\mathrm{reg}}},$$

where the composition is the identity. Using that the involved sheaves are reflexive, we obtain a nonzero global section of $F_*\mathcal{O}_X((1-p)K_X)$. This gives a nonzero element of $H^0(X,\mathcal{O}_X((1-p)K_X))$, hence $(1-p)K_X$ is effective. If no positive multiple of K_X is effective, then $\kappa(X,K_X)=-\infty$. So assume that nK_X is effective for some $n>0$. Since $n(p-1)K_X$ and $n(1-p)K_X$ are both effective, $n(1-p)K_X$ is trivial by Lemma 2.1. □

Remark 8.3. Note that F-split varieties may have trivial canonical divisor. For instance, ordinary elliptic curves and ordinary $K3$ or abelian surfaces are F-split ; see [BK05, Rmk. 7.5.3(i)].

In order to prove Theorem 8.1, it remains to show that the canonical divisor of a proper splinter is not torsion. First we note that the Picard group of a proper splinter is torsion-free ; this is a small generalization of a result of Carvajal-Rojas [CR22, Cor. 5.4] who showed that a globally F-regular projective variety has torsion-free Picard group. In particular, this provides a proof of Theorem 8.1 if K_X is Cartier, e.g., if X is in addition assumed to be Gorenstein.

Proposition 8.4. *Let X be a proper scheme over a field of positive characteristic. If X is a splinter, then* $\mathrm{Pic}(X)$ *is torsion-free.*

Proof. We argue as in [CR22, Rmk. 5.6] which is concerned with the globally F-regular case. If $\mathcal{L}$ is a torsion line bundle, then it is in particular semiample and therefore if X is a splinter, then we have $\chi(X,\mathcal{L})=h^0(X,\mathcal{L})$ by [Bha12, Prop. 7.2] (recalled in Proposition 3.6). But then $0=\chi(X,\mathcal{L})=\chi(X,\mathcal{O}_X)=1$ if $\mathcal{L}$ is nontrivial. This is impossible.

Alternately, one can argue using Theorem 7.8 as follows. An n-torsion line bundle $\mathcal{L}$ on X gives rise to a nontrivial μ_n-torsor $\pi\colon Y\to X$, where Y is defined to be the relative spectrum of the finite $\mathcal{O}_X$-algebra $\mathcal{O}_X\oplus\mathcal{L}\oplus\cdots\oplus\mathcal{L}^{n-1}$. If $n>1$, we must have $H^0(X,\mathcal{L})=0$, since any nontrivial section $s\colon\mathcal{O}_X\to\mathcal{L}$ would also give a nontrivial section s^n of $\mathcal{O}_X$, so s would be nowhere vanishing and therefore $\mathcal{L}$ would be trivial. This yields the equality

$$H^0(Y,\mathcal{O}_Y)=H^0(X,\pi_*\mathcal{O}_Y)=H^0(X,\mathcal{O}_X\oplus\mathcal{L}\oplus\cdots\oplus\mathcal{L}^{n-1})=H^0(X,\mathcal{O}_X).$$

We conclude from Theorem 7.8 that if X is a splinter, then $\deg(\pi)=1$, i.e., $n=1$ and $\mathcal{L}$ is trivial. □

To deal with the non-Gorenstein case, we have :

Proposition 8.5. *Let X be a connected positive-dimensional proper scheme over a field of positive characteristic. If X is a splinter, then the canonical divisor K_X is not torsion.*

Proof. Assume for contradiction that K_X is torsion of order r, i.e., that $\omega_X|_{X_{\mathrm{reg}}}$ is a torsion line bundle of order r. By considering the relative spectrum, we obtain a μ_r-quasi-torsor

$$\pi\colon Y=\mathrm{Spec}_X\left(\bigoplus_{i=0}^{r-1}\mathcal{O}_X(iK_X)\right)\to X,$$

which, over X_{reg}, restricts to a μ_r-torsor $\pi|_U\colon V:=\pi^{-1}(X_{\mathrm{reg}})\to X_{\mathrm{reg}}=:U$.

In addition, $\pi_*\mathcal{O}_Y=\bigoplus_{i=0}^{r-1}\mathcal{O}_X(iK_X)$, and by Lemma 2.1 the sheaves $\mathcal{O}_X(iK_X)$ have no nonzero global sections for $1\le i\le r-1$. Thus $H^0(V,\mathcal{O}_V)=H^0(X_{\mathrm{reg}},\mathcal{O}_{X_{\mathrm{reg}}})=H^0(X,\mathcal{O}_X)$, where the second equality holds by normality of X, is a field and we conclude as in Proposition 7.4 that V is a splinter. In particular, V is normal and therefore, by, e.g., [Sta23, Tag 035K & Tag 035E], the normalization $Y^\nu\to Y$ is an isomorphism over V. Since normalization is finite, $Y^\nu\setminus V$ has codimension ≥ 2 and thus Y^ν is a splinter by Lemma 4.1.

Now $\pi_U^! \mathcal{O}_{X_{\mathrm{reg}}} = \mathcal{O}_V$ holds by Lemma 7.1 and this implies $\pi_U^* \omega_{X_{\mathrm{reg}}} = \omega_V$. On the other hand, we have isomorphisms

$$(\pi_U)_* \pi_U^* \omega_{X_{\mathrm{reg}}} \cong \omega_{X_{\mathrm{reg}}} \otimes_{\mathcal{O}_{X_{\mathrm{reg}}}} \bigoplus_{i=0}^{r-1} \omega_{X_{\mathrm{reg}}}^i = \bigoplus_{i=1}^{r} \omega_{X_{\mathrm{reg}}}^i \cong \bigoplus_{i=0}^{r-1} \omega_{X_{\mathrm{reg}}}^i \cong (\pi_U)_* \mathcal{O}_V$$

as $(\pi_U)_* \mathcal{O}_V$-modules. Hence $V = \mathrm{Spec}_{X_{\mathrm{reg}}} \bigoplus_{i=0}^{r-1} \omega_{X_{\mathrm{reg}}}^i$ has trivial dualizing sheaf. Consequently, since ω_{Y^ν} is a reflexive sheaf, Y^ν has trivial dualizing sheaf. But, by Lemma 3.7, a proper splinter cannot have trivial canonical sheaf. $\square$

Proof of Theorem 8.1. The canonical divisor K_X is not torsion by Proposition 8.5 (or more simply by Proposition 8.4 if K_X is Cartier, e.g. if X is Gorenstein), and it follows from Lemma 8.2 that $\kappa(X, K_X) = -\infty$. $\square$

Remark 8.6 (Torsion Weil divisors on splinters). Proper splinters may have nontrivial torsion Weil divisor classes. Indeed, projective toric varieties are globally F-regular and in particular splinters, and Carvajal-Rojas [CR22, Ex. 5.7] gives an example of a projective toric surface that admits a nontrivial 2-torsion Weil divisor class.

9. Vanishing of global differential forms

Fix a perfect field k of positive characteristic p and let X be a smooth proper variety over k. Let $\Omega^\bullet_{X/k}$ be the de Rham complex and recall, e.g. from [Kat70, Thm. 7.2], that there exists an isomorphism of graded $\mathcal{O}_X$-modules

$$C_X^{-1} \colon \bigoplus_{j \geq 0} \Omega_X^j \to \bigoplus_{j \geq 0} \mathcal{H}^j(F_* \Omega_X^\bullet),$$

whose inverse C_X is the so-called *Cartier operator*. It gives rise for all $j \geq 0$ to short exact sequences of $\mathcal{O}_X$-modules

$$0 \to B_X^j \to Z_X^j \xrightarrow{C_X^j} \Omega_X^j \to 0, \tag{9.1}$$

where B_X^j denotes the j-th coboundaries and Z_X^j the j-th cocycles of $F_*\Omega_X^\bullet$. Note that these coincide with the image under F_* of the coboundaries and cocycles of $\Omega_X^\bullet$. Moreover, there is a short exact sequence

$$0 \to \mathcal{O}_X \to F_* \mathcal{O}_X \to B_X^1 \to 0.$$

The proof of the following theorem is inspired by the proof of [AWZ21, Lem. 6.3.1].

Theorem 9.2 (Theorem **(C)**)**.** *Let X be a smooth proper variety over a field k of positive characteristic p. If X is a splinter, then $H^0(X, \Omega_X^1) = 0$.*

Proof. Clearly, we may and do assume that X is connected. Let $\bar{k}$ be an algebraic closure of k. It is enough to show that $H^0(X_{\bar{k}}, \Omega^1_{X_{\bar{k}}}) = 0$. Since X is in particular geometrically reduced over k, $H^0(X, \mathcal{O}_X)$ is a finite separable extension of k ; see, e.g., [Sta23, Tag 0BUG]. It follows that $X_{\bar{k}}$ is the disjoint union of $\dim_k H^0(X, \mathcal{O}_X)$ copies of $X \times_{H^0(X, \mathcal{O}_X)} \bar{k}$. From Proposition 5.9, we find that $X_{\bar{k}}$ is a splinter. Therefore it is enough to establish the theorem in case k is algebraically closed. So assume k is algebraically closed, in which case the p-th power map $k = H^0(X, \mathcal{O}_X) \to H^0(X, F_* \mathcal{O}_X) = k$ is an isomorphism. Since by Proposition 3.6, for a splinter X, $\mathcal{O}_X$ has zero cohomology in positive degrees, the long exact sequence associated to

$$0 \to \mathcal{O}_X \to F_* \mathcal{O}_X \to B_X^1 \to 0$$

gives that $H^j(X, B_X^1) = 0$ for all $j \geq 0$. From the long exact sequence associated to (9.1), we find that the Cartier operator induces an isomorphism $H^0(X, Z_X^1) \to H^0(X, \Omega_X^1)$. In particular, $\dim H^0(X, Z_X^1) = \dim H^0(X, \Omega_X^1)$. The inclusion of closed 1-forms $\ker(d \colon \Omega_X^1 \to \Omega_X^2) \subseteq \Omega_X^1$ yields an injection $H^0(X, \ker(d \colon \Omega_X^1 \to \Omega_X^2)) \subseteq H^0(X, \Omega_X^1)$. Since $H^0(X, \ker(d \colon \Omega_X^1 \to \Omega_X^2)) = H^0(X, Z_X^1)$, the above inclusion is in fact an equality. In other words, any global 1-form on X is closed. By [GK03, Prop. 4.3], there exists an isomorphism

$$H^0(X, \Omega_X^1) \cong \mathrm{Pic}(X)[p] \otimes_{\mathbb{Z}} k,$$

where $\mathrm{Pic}(X)[p]$ denotes the p-torsion line bundles on X. By Proposition 8.4, $\mathrm{Pic}(X)$ is torsion-free. □

10. On the splinter property for proper surfaces

A proper curve over an algebraically closed field of positive characteristic is a splinter if and only if it is isomorphic to the projective line. In this section, we investigate which proper surfaces over an algebraically closed field of positive characteristic are splinters. First we show in Proposition 10.1 that proper surface splinters are rational. We then show in Proposition 10.4 that the blow-up of the projective plane in any number of closed points lying on a given conic is a splinter. On the other hand, we give examples of rational surfaces that are not splinters in Section 10.3.

10.1. **Proper splinter surfaces are rational.** The fact proved in Theorem 8.1 that proper splinters in positive characteristic have negative Kodaira dimension can be used to show that proper surface splinters over an algebraically closed field of positive characteristic are rational:

Proposition 10.1. *Let X be an irreducible proper surface over an algebraically closed field of positive characteristic. If X is a splinter, then X is rational.*

Proof. If X is not smooth, choose a resolution of singularities $\pi\colon \tilde{X} \to X$ such that π is an isomorphism over the regular locus X_{reg}, which exists by [Lip69, §2]. It suffices to show that $\tilde{X}$ is rational. Note that Grauert–Riemenschneider vanishing holds for surfaces, see, e.g., [Sta23, Tag 0AXD]. Thus, the proof of [Bha12, Thm. 2.12] shows that the above resolution of singularities satisfies $\mathrm{R}\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$, that is, that X has rational singularities. Therefore

$$\chi(\mathcal{O}_{\tilde{X}}) = \chi(\mathrm{R}\pi_*\mathcal{O}_{\tilde{X}}) = \chi(\mathcal{O}_X) = 1.$$

By Castelnuovo's rationality criterion it remains to show that $\omega_{\tilde{X}}^2$ has no nonzero global sections. So assume that there is a nonzero section $s \in H^0(\tilde{X}, \omega_{\tilde{X}}^2)$. Then s is nonzero after restriction to the open subset $\pi^{-1}(X_{\mathrm{reg}})$. Since π is an isomorphism over X_{reg}, the section s would provide a nonzero section of $\mathcal{O}_X(2K_X)$, contradicting Theorem 8.1. □

10.2. **Examples of projective rational surfaces that are splinters.** We give examples of projective rational surfaces that are splinters; in all cases, this is achieved by showing that they are globally F-regular. We start with already known examples.

Example 10.2 (Del Pezzo surfaces, [Har98, Ex. 5.5])**.** Let X be a smooth projective del Pezzo surface over an algebraically closed field of characteristic $p > 0$. Then X is globally F-regular if one of the following conditions holds:

(i) $K_X^2 > 3$,
(ii) $K_X^2 = 3$ and $p > 2$,
(iii) $K_X^2 = 2$ and $p > 3$, or
(iv) $K_X^2 = 1$ and $p > 5$.

Moreover, if none of the above conditions are satisfied, there are globally F-regular and non globally F-regular cases; for instance, the Fermat cubic threefold in characteristic 2 is not globally F-regular.

Example 10.3 (Hirzebruch surfaces, [GT16, Prop. 3.1])**.** If X is a Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ over a perfect field of positive characteristic, then X is globally F-regular.

To the above list of examples, we can add blow-ups of $\mathbb{P}^2$ in any number of points lying on a conic:

Proposition 10.4. *Let k be an algebraically closed field of characteristic $p > 0$ and let $p_1, \dots, p_n$ be distinct closed points in $\mathbb{P}^2_k$. Assume either of the following conditions:*

(i) *The points $p_1, \dots, p_n$ lie on a line $L \subseteq \mathbb{P}^2_k$.*
(ii) *The points $p_1, \dots, p_n$ lie on a (possibly singular) conic $C \subseteq \mathbb{P}^2_k$.*

then the blow-up $X := \mathrm{Bl}_{\{p_1,\dots,p_n\}}\mathbb{P}^2_k$ is globally F-regular. In particular, the blow-up of $\mathbb{P}^2_k$ in at most 5 closed points is globally F-regular.

In some sense Proposition 10.4 is optimal since by Example 10.2 the Fermat cubic surface, which is the blow-up of $\mathbb{P}^2_k$ in 6 points, is not globally F-regular if $\operatorname{char} k = 2$. The proof adapts a strategy which is similar to the arguments of [BK05, §§1.3-1.4]. First we determine, under the isomorphism $\operatorname{Hom}_X(F_*\mathcal{O}_X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X((1-p)K_X))$, which global sections correspond to splittings of $\mathcal{O}_X \to F_*\mathcal{O}_X$ in the case where $X = \mathbb{P}^n_k$. For the sake of completeness, we state and prove the following lemma, which appears as an exercise in [BK05].

Lemma 10.5 (cf. [BK05, Ex. 1.3.E(1)])**.** *Let $X = \mathbb{P}^n_k = \operatorname{Proj} k[X_0, \dots, X_n]$ for a field k of characteristic $p > 0$. Let $\eta\colon \mathcal{O}_X \to F^e_*\mathcal{O}_X$ be the canonical map and consider the following chain of isomorphisms*

$$\Phi\colon \operatorname{Hom}_X(F^e_*\mathcal{O}_X, \mathcal{O}_X) \xrightarrow{\cong} \operatorname{Ext}^n_X(\mathcal{O}_X, (F^e_*\mathcal{O}_X) \otimes \omega_X)^\vee \xrightarrow{\cong} H^n(X, \omega_X^{p^e})^\vee \xrightarrow{\cong} H^0(X, \omega_X^{1-p^e})$$

where the first and last isomorphism are given by Serre duality and the second isomorphism follows from the projection formula. Then the following diagram commutes

$$\begin{tikzcd} \operatorname{Hom}_X(F^e_*\mathcal{O}_X, \mathcal{O}_X) \arrow[r, "-\circ\eta"] \arrow[d, "\Phi"] & \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_X) \arrow[d, "\mathrm{ev}_1"] \\ H^0(X, \mathcal{O}_X(\omega_X^{1-p^e})) \arrow[r, "\tau"] & k, \end{tikzcd}$$

where ev_1 is the evaluation at the constant global section 1 and τ is the map sending a homogeneous polynomial $P \in H^0(X, \mathcal{O}_X((n+1)(p^e-1)))$ of degree $(n+1)(p^e-1)$ to the coefficient of the monomial $(X_0 \cdots X_n)^{(p^e-1)}$ in P. In particular, $\Phi^{-1}(P)$ provides a splitting of η if and only if $\tau(P) = 1$.

Proof. Recall, e.g. from [Har77, Ch. III, Thm. 5.1 & Thm. 7.1], that Serre duality for line bundles on $\mathbb{P}^n$ is given by the bilinear form

$$\begin{aligned} H^0(X, \mathcal{O}_X(a)) \otimes H^n(X, \mathcal{O}_X(-(n+1)-a)) &\to H^n(X, \mathcal{O}_X(-(n+1))) = k\frac{1}{X_0 \cdots X_n} \\ (P, Q) &\mapsto \text{coefficient of } (X_0 \cdots X_n)^{-1} \text{ in } PQ, \end{aligned}$$

where, for $b < 0$, we identify $H^n(X, \mathcal{O}_X(b))$ with the degree b part of the negatively graded k-algebra $(X_0 \cdots X_n)^{-1}k[X_0^{-1}, \dots, X_n^{-1}]$. Consider the following commutative diagram of isomorphisms

$$\begin{tikzcd} & \operatorname{Hom}_X(F^e_*\mathcal{O}_X, \mathcal{O}_X) \arrow[r, "-\circ\eta"] \arrow[d, "\mathrm{SD}"] & \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_X) \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\mathrm{SD}"] & k \\ H^0(X, \mathcal{O}_X((1-p^e)K_X)) \arrow[r, "\mathrm{SD}"] & H^n(X, \mathcal{O}_X(p^eK_X))^\vee \arrow[r, "(F^{e*})^\vee"] & H^n(X, \mathcal{O}_X(K_X))^\vee, & \end{tikzcd}$$

where SD stands for Serre duality. Since $F^{e*}\colon H^n(X, \mathcal{O}_X(-(n+1))) \to H^n(X, \mathcal{O}_X(-p^e(n+1)))$ raises a polynomial to its p^e-th power, we find that a monomial in $H^0(X, \mathcal{O}_X((1-p^e)K_X))$ is sent to 1 in k along the above diagram if it is $(X_0 \cdots X_n)^{p^e-1}$ and to zero otherwise. $\square$

For X the blow-up of $\mathbb{P}^2_k$ in n distinct closed points $p_1, \dots, p_n \in \mathbb{P}^2_k$, we denote by E_i the exceptional curve over the point p_i and we let $H \in \operatorname{Pic}(X)$ be the pullback of the class of a hyperplane in $\mathbb{P}^2_k$. Then

$$\operatorname{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_n,$$

is an orthogonal decomposition with respect to the intersection pairing, and we have $H^2 = 1$ and $E_i^2 = -1$ for all $1 \le i \le n$. The canonical class of X is $K_X = -3H + \sum_i E_i$. If $C \subseteq \mathbb{P}^2_k$ is an irreducible curve, its strict transform $\tilde{C} \subseteq X$ has class

$$\tilde{C} = dH - \sum_{i=1}^n m_iE_i \in \operatorname{Pic}(X),$$

where d is the degree of C and m_i is the multiplicity of C at p_i. Any irreducible curve in X is either one of the exceptional curves E_i, or the strict transform of an irreducible curve in $\mathbb{P}^2_k$. Note that for $d > n$, the divisor $dH - \sum_i E_i$ is ample, since it has positive square and since the intersection with any integral curve in X is positive.

Proof of Proposition 10.4. Since X is smooth, it suffices by Theorem 6.1 (see Remark 6.2) to prove that there exists an ample divisor D on X such that $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ splits for some $e > 0$.

Let $d > 0$ be an integer such that the divisor $D := dH - \sum_i E_i$ on X is ample and fix a global section $\sigma\colon \mathcal{O}_X \to \mathcal{O}_X(D)$. We can interpret σ as a homogeneous polynomial of degree d vanishing at the points $p_1, \dots, p_n$. Now, as in Lemma 10.5, we have an isomorphism

$$\Psi\colon \operatorname{Hom}_X(F^e_*\mathcal{O}_X(D), \mathcal{O}_X) \xrightarrow{\cong} H^0(X, \mathcal{O}_X((1-p^e)K_X - D))$$

and global sections of $\mathcal{O}_X((1-p^e)K_X - D)$ correspond again to polynomials of a certain degree, vanishing to some certain order at the points p_i. The following claim is similar to [BK05, p. 39].

Claim. A section $\varphi \in H^0(X, \mathcal{O}_X((1-p^e)K_X - D))$ defines a section of $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ if and only if $\Psi^{-1}(\varphi)\sigma \in H^0(X, \mathcal{O}_X((1-p^e)K_X))$ defines a splitting of $\mathcal{O}_X \to F^e_*\mathcal{O}_X$, where $\Psi^{-1}(\varphi)\sigma$ is the usual product of polynomials.

Proof of the claim. A map $\varphi \in \operatorname{Hom}_X(F^e_*\mathcal{O}_X(D), \mathcal{O}_X)$ is a section of $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ if and only if $\varphi \circ F^e_*(\sigma)$ is a section of $\mathcal{O}_X \to F^e_*\mathcal{O}_X$. Thus, we have to check that the composition $\varphi \circ F^e_*(\sigma)$ corresponds to the product $\Psi(\varphi)\sigma$. This is done by verifying, that the following diagram commutes

$$\begin{tikzcd}
\operatorname{Hom}_X(F^e_*\mathcal{O}_X(D), \mathcal{O}_X) \arrow[r, "- \circ F^e_*(\sigma)"] \arrow[d, "\mathrm{SD}"] & \operatorname{Hom}_X(F^e_*\mathcal{O}_X, \mathcal{O}_X) \arrow[d, "\mathrm{SD}"] \\
H^n(X, \omega_X \otimes F^e_*\mathcal{O}_X(D))^\vee \arrow[r, "(\omega_X \otimes F^e_*(\sigma))^\vee"] \arrow[d, "\text{projection formula}"] & H^n(X, \omega_X \otimes F^e_*\mathcal{O}_X)^\vee \arrow[d, "\text{projection formula}"] \\
H^n(X, F^e_*\mathcal{O}_X(p^eK_X + D))^\vee \arrow[r, "(F^e_*(\mathcal{O}_X(p^eK_X) \otimes \sigma))^\vee"] \arrow[d] & H^n(X, F^e_*\mathcal{O}_X(p^eK_X))^\vee \arrow[d] \\
H^n(X, \mathcal{O}_X(p^eK_X + D))^\vee \arrow[r, "(\mathcal{O}_X(p^eK_X) \otimes \sigma)^\vee"] \arrow[d, "\mathrm{SD}"] & H^n(X, \mathcal{O}_X(p^eK_X))^\vee \arrow[d, "\mathrm{SD}"] \\
H^0(X, \mathcal{O}_X((1-p^e)K_X - D)) \arrow[r, "- \cdot \sigma"] & H^0(X, \mathcal{O}_X((1-p^e)K_X)).
\end{tikzcd}$$ $\square$

Denote by $\mu\colon X \to \mathbb{P}^2$ the blow-up map. Since $\mu_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2}$, μ_* induces an isomorphism $\operatorname{End}(\mathcal{O}_X) \to \operatorname{End}(\mathcal{O}_{\mathbb{P}^2})$. Since any morphism of schemes commutes with the Frobenius, a map $\varphi\colon F^e_*\mathcal{O}_X \to \mathcal{O}_X$ is a splitting of $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ if and only if $\mu_*(\varphi)$ is a splitting of $\mathcal{O}_{\mathbb{P}^2} \to F^e_*\mathcal{O}_{\mathbb{P}^2}$. The proposition will follow if we can find suitable polynomials $\varphi \in H^0(X, \mathcal{O}_X((1-p^e)K_X - D))$ and $\sigma \in H^0(X, \mathcal{O}_X(D))$ such that $\mu_*(\varphi\sigma)$ defines a splitting of $\mathcal{O}_{\mathbb{P}^2} \to F^e_*\mathcal{O}_{\mathbb{P}^2}$. In terms of Lemma 10.5, the monomial $(XYZ)^{p^e-1}$ has to occur with coefficient 1 in $\varphi\sigma$ (here we are using coordinates $\mathbb{P}^2_k = \operatorname{Proj} k[X, Y, Z]$).

We first compute

$$(1-p^e)K_X - D = 3(p^e-1)H - (p^e-1)\sum_i E_i - dH + \sum_i E_i = (3(p^e-1) - d)H - (p^e-2)\sum_i E_i.$$

If all the points p_i lie on a line L, we can assume without loss of generality that $L = V(Z)$. Consider the polynomials $\tilde{\varphi} := X^{(p^e-1)-(d-1)}Y^{p^e-1}$ and $\tilde{\sigma} := X^{d-1}$. Moreover, if we set $\varphi := \tilde{\varphi}Z^{p^e-2}$ and $\sigma := \tilde{\sigma}Z$, then $\varphi \in H^0(X, \mathcal{O}_X((1-p^e)K_X - D))$ and $\sigma \in H^0(X, \mathcal{O}_X(D))$ and the coefficient of $(XYZ)^{p^e-1}$ in $\varphi\sigma$ is 1.

If the points lie on a conic C, we can assume after possible change of coordinates that the conic is given by an equation of the form $XY - Z^2$, XY, or X^2; see the elementary Lemma 10.7 below. In the last case, the points lie on the line $X = 0$, thus we may assume that C is given by one of the equations $\tilde{\sigma} = XY - Z^2$ or $\tilde{\sigma} = XY$. Now set $\sigma := Z^{d-2}\tilde{\sigma}$ and $\varphi := Z^{(p^e-1)-(d-2)}\tilde{\sigma}^{p^e-2}$ and observe that $(XYZ)^{p^e-1}$ occurs with coefficient 1 in $\varphi\sigma$. $\square$

Remark 10.6. Recall, e.g. from [Har77, Ch. IV, Prop. 4.21], that an elliptic curve $C = V(P) \subseteq \mathbb{P}^2_k$ is *ordinary* if $(XYZ)^{p-1}$ occurs with nonzero coeffcient in P^{p-1}. The above Lemma 10.5 can also be used to show, similarly to [Har15, Rmk. 6.3], that the blow-up of $\mathbb{P}^2_k$ in any number of points, which lie on an ordinary elliptic curve, is F-split.

Lemma 10.7. *Let k be an algebraically closed field. After suitable coordinate transform a conic in $\mathbb{P}^2_k$ is given one of the following equations:*

$$XY - Z^2, \ XY, \ \textit{or}\ X^2.$$

Proof. First recall that $\mathrm{PGL}_3(k)$ acts 2-transitively on $\mathbb{P}^2_k = \operatorname{Proj} k[X,Y,Z]$. In particular, $\mathrm{PGL}_3(k)$ acts also 2-transitively on the set of lines in $\mathbb{P}^2_k$, which is $\mathbb{P}(H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(1))) = \mathbb{P}(k[X,Y,Z]_{\deg 1})$. If a conic $C \subseteq \mathbb{P}^2_k$ is reducible, then it is either a double line or the union of two lines and we can assume C to be defined by the equations $X^2 = 0$ or $XY = 0$, resp. It remains to show that an irreducible conic C is defined by the equation $XY - Z^2 = 0$ after suitable coordinate transformation. We follow the arguments of [Kir92, Cor. 3.12]. Since C has only finitely many singular points, we can assume without loss of generality that $[0:1:0]$ is a smooth point of C and the line $Z = 0$ is tangent to C at $[0:1:0]$. This means that C is given by an equation of the form

$$aYZ + bX^2 + cXZ + dZ^2, \tag{10.8}$$

since the line tangent to C at the point $[0:1:0]$ is precisely the line given by the linear factors in the dehomogenized equation defining C on the affine open $\{Y \neq 0\} \cong \operatorname{Spec} k[X,Z]$. By assumption (10.8) is an irreducible polynomial. This implies that $b \neq 0$ and $a \neq 0$. We conclude by noting that $XY - Z^2$ is mapped to (10.8) under the coordinate transformation

$$[x,y,z] \mapsto [\sqrt{b}x, ay + cx + dz, -z]. \qquad \square$$

10.3. **Examples of projective rational surfaces that are not splinters.** In this paragraph, we use Bhatt's Proposition 3.6 to show that certain projective rational surfaces are not splinters. First, we have the following example of surfaces that are not globally F-regular.

Example 10.9 ([SS10, Ex. 6.6])**.** Let k be an algebraically closed field of positive characteristic. If X is the blow-up of $\mathbb{P}^2_k$ in 9 closed points in general position, then $-K_X$ is not big. Therefore, by Proposition 3.11, X is not globally F-regular. Furthermore, this shows that the blow-up of $\mathbb{P}^2$ in at least 9 points in general position is not globally F-regular, as global F-regularity descends along birational morphisms (Lemma 6.3).
On the other hand, we note that if X is the blow-up of $\mathbb{P}^2$ in 9 points lying on an ordinary elliptic curve, then X is F-split; see Remark 10.6. Since being ordinary is an open property, it follows that the blow-up of $\mathbb{P}^2$ in 9 points in general position is F-split.

We can extend Example 10.9 and show that in some cases the blow-up of $\mathbb{P}^2$ in 9 points is not a splinter:

Proposition 10.10. *Let X be the blow-up of $\mathbb{P}^2_k$ in 9 distinct k-rational points. Assume either of the following conditions:*

(i) *The base field k is the algebraic closure of a finite field and the 9 points lie on a smooth cubic curve (e.g., the 9 points are in general position).*
(ii) *The base field k has positive characteristic and the 9 points lie at the transverse intersection of two cubic curves in $\mathbb{P}^2_k$.*

Then X is not a splinter.

Proof. In both cases, we show that the anticanonical line bundle ω_X^{-1} is semiample and satisfies $H^1(X, \omega_X^{-1}) \neq 0$. It follows from Proposition 3.6 that X is not a splinter.

In case (i), the anticanonical divisor $-K_X$ is the strict transform of the smooth cubic curve and is therefore smooth of genus 1. Since $(-K_X)^2 = 0$, we get from [Tot09, Thm. 2.1] that $-K_X$ is semiample, that is, there exists a positive integer n such that $-nK_X$ is basepoint free. In particular, since K_X is not torsion, $h^0(-nK_X) \geq 2$. By Riemann–Roch $\chi(-nK_X) = 1$, and hence $h^1(-nK_X) \neq 0$.

In case (ii), $-K_X$ is basepoint free (in particular semiample) and satisfies $h^0(-K_X) \geq 2$. Indeed, $-K_X$ admits two sections, corresponding to the strict transforms of the cubics, which do not meet in any point in the blow-up as they pass through the 9 points in $\mathbb{P}^2_k$ from different tangent directions. On the other hand, we have $\chi(-K_X) = 1$ and it follows that $h^1(-K_X) \neq 0$. $\square$

Remark 10.11. An alternative proof of Proposition 10.10(ii) can be obtained by using Lemma 4.3. Indeed, if the 9 blown up points lie in the intersection of two distinct smooth cubic curves, the set of cubic curves passing through the 9 points forms a pencil and we obtain an elliptic fibration $X \to \mathbb{P}^1$. Since the generic fiber of this fibration is not a splinter, X is not a splinter.

Finally, further examples of rational surfaces over $\overline{\mathbb{F}}_p$ that are not splinters are provided by the following:

Proposition 10.12. *Let k be the algebraic closure of a finite field. Let $d \geq 4$ be an integer and let C be an irreducible curve of degree d in $\mathbb{P}^2_k$. Let $n := \binom{2+d}{2} = \frac{(d+2)(d+1)}{2}$; e.g., $n = 15$ if $d = 4$. The blow-up X of $\mathbb{P}^2_k$ in n distinct smooth points of C is not a splinter.*

Proof. By Proposition 3.6, it suffices to construct a semiample line bundle $\mathcal{L}$ on X such that $H^1(X, \mathcal{L}) \neq 0$. For that purpose, we consider the class D of the strict transform of the curve C. We have $D = dH - \sum_{i=1}^n E_i$, where H is the pullback of the hyperplane class in $\mathbb{P}^2_k$ and E_i are the exceptional curves lying above the n blown up points. We claim that the line bundle $\mathcal{L} := \mathcal{O}_X(D)$ is semiample and satisfies $H^1(X, \mathcal{L}) \neq 0$. On the one hand, D is nef since it is effective and satisfies $D^2 = d^2 - n > 0$ for $d \geq 4$. Being nef and having positive self intersection, it is also big, see, e.g., [Kol96, Cor. 2.16]. By Keel's [Kee99, Cor. 0.3] any nef and big line bundle on a surface over the algebraic closure of a finite field is semiample. On the other hand, by Riemann–Roch

$$\chi(D) = 1 + \frac{1}{2}(D^2 - K_X \cdot D) = 1 + \frac{1}{2}(d^2 + 3d - 2n) = 0,$$

for $K_X = -3H + \sum_i E_i$ the canonical divisor on X. Since D is effective, $H^0(X, \mathcal{L}) \neq 0$, and we conclude that $H^1(X, \mathcal{L}) \neq 0$. $\square$

11. K-equivalence, $\mathcal{O}$-equivalence, and D-equivalence

The aim of this section is to study the derived-invariance of the (derived) splinter property and of global F-regularity for projective varieties over a field of positive characteristic. For that purpose, we introduce the notion of $\mathcal{O}$-equivalence, which is closely related to K-equivalence but offers more flexibility, and show that both, the (derived) splinter property and global F-regularity, are preserved under $\mathcal{O}$-equivalence.

11.1. **K-equivalence.** In this paragraph, we fix an excellent Noetherian scheme S admitting a dualizing complex $\omega_S^\bullet$. Any scheme X over S with structure morphism $h\colon X \to S$ of finite type and separated will be endowed with the dualizing complex $\omega_X^\bullet := h^!\omega_S^\bullet$ [Sta23, Tag 0AU3]. If X is normal, there is a unique, up to linear equivalence, Weil divisor K_X on X such that $\omega_X \cong \mathcal{O}_X(K_X)$. The following notions are classical, at least in the case of smooth varieties over a field (where they agree):

Definition 11.1 (K-equivalence and strong K-equivalence)**.** Let X and Y be integral normal $\mathbb{Q}$-Gorenstein schemes of finite type and separated over S. We say X and Y are *K-equivalent* if there exists a normal scheme Z over S with proper birational S-morphisms $p\colon Z \to X$ and $q\colon Z \to Y$ such that p^*K_X and q^*K_Y are $\mathbb{Q}$-linearly equivalent.

If in addition X and Y are Gorenstein, we say X and Y are *strongly K-equivalent* if there exists a normal scheme Z over S with proper birational S-morphisms $p\colon Z \to X$ and $q\colon Z \to Y$ such that p^*K_X and q^*K_Y are linearly equivalent.

Obviously, if X and Y are Gorenstein and strongly K-equivalent, then they are K-equivalent. The converse holds provided X and Y are not too singular; see Remark 11.3 below. The following Proposition 11.2 says, in particular, that K-equivalent normal terminal varieties are isomorphic in codimension 1. This is certainly well-known, at least in characteristic zero, see, e.g., [Kaw02, Lem. 4.2], but as we were not able to find a suitable reference for our more general setting, we provide a proof. An integral normal excellent scheme X of finite type and separated over S is said to be *terminal* (resp. *canonical*) if X is $\mathbb{Q}$-Gorenstein and for any proper surjective S-morphism $f\colon X' \to X$, the discrepancies of the exceptional divisors are all positive (resp. nonnegative).

Proposition 11.2. *Let X and Y be integral normal terminal schemes of finite type and separated over S. If X and Y are K-equivalent, then the induced birational map $X \dashrightarrow Y$ is small in the sense of Definition 4.6.*

Proof. Let Z be a normal scheme over S with proper birational morphisms $p\colon Z \to X$ and $q\colon Z \to Y$ such that $p^*K_X \sim_{\mathbb{Q}} q^*K_Y$. Since X and Y are terminal, we have

$$p^*K_X + \sum a_i E_i \sim_{\mathbb{Q}} K_Z \sim_{\mathbb{Q}} q^*K_Y + \sum b_j F_j,$$

where $a_i, b_j \in \mathbb{Q}_{>0}$ and $E_i \subseteq \operatorname{Exc}(p)$, $F_j \subseteq \operatorname{Exc}(q)$ are the irreducible components of codimension 1 endowed with their reduced structure. Thus, we have linearly equivalent effective $\mathbb{Q}$-divisors $K_{Z/X} = \sum_i a_i E_i$ and $K_{Z/Y} = \sum_j b_j F_j$ such that $\operatorname{Supp}(K_{Z/X}) = \operatorname{Exc}(p)$ and $\operatorname{Supp}(K_{Z/Y}) = \operatorname{Exc}(q)$ up to some locally closed subsets of codimension ≥ 2. Let

$$D \coloneqq p^*K_X - q^*K_Y = K_{Z/Y} - K_{Z/X} = \sum b_j F_j - \sum a_i E_i.$$

We have $D \sim_{\mathbb{Q}} 0$, and since X and Y are $\mathbb{Q}$-Gorenstein, the divisor D is $\mathbb{Q}$-Cartier. Note that $-D$ is p-nef and $p_*D = \sum_j b_j p_* F_j$ is effective. By the Negativity Lemma [Bha+22, Lem. 2.16] (which can be applied since S is assumed to be excellent and since we can assume that p and q are projective after applying Chow's Lemma [Sta23, Tag 02O2] and normalizing [Sta23, Tag 035E]), D is effective. (For the classical version of the Negativity Lemma in characteristic zero, see [KM98, Lem. 3.39].) Arguing similarly for $-D$ shows that $-D$ is effective, and therefore that $D = 0$. Hence, $\operatorname{Exc}(p) = \operatorname{Exc}(q)$ up to some locally closed subsets of codimension ≥ 2. This proves the statement, since $p(\operatorname{Exc}(p))$ and $q(\operatorname{Exc}(q))$ both have codimension ≥ 2 (a proper birational morphism to a normal Noetherian scheme has geometrically connected fibers by [Sta23, Tag 03H0]). □

Remark 11.3. Let X and Y be integral normal Gorenstein canonical schemes of finite type and separated over S. Employing the Negativity Lemma as in the proof of Proposition 11.2 shows that if X and Y are K-equivalent, then they are strongly K-equivalent.

Together with Lemma 4.1, we obtain that both the splinter property and global F-regularity for normal terminal varieties over a field of positive characteristic are invariant under K-equivalence:

Corollary 11.4. *Let X and Y be K-equivalent integral normal terminal schemes of finite type and separated over S admitting a dualizing complex. The following statements hold.*

(i) *X is a splinter if and only if Y is a splinter.*
(ii) *X is globally F-regular if and only if Y is globally F-regular, provided $S = \operatorname{Spec} k$ with k an F-finite field.*

Proof. This is the combination of Lemma 4.4, Proposition 4.7, and Proposition 11.2. □

Remark 11.5. Proposition 11.2 does not hold without restrictions on the singularities of X and Y. Consider indeed any crepant morphism $p\colon Y \to X$ to a Gorenstein variety X over a field k, with exceptional locus containing a divisor. Then $Y = Z \to X$ provides a strong K-equivalence between the Gorenstein proper varieties X and Y that does not induce a small birational map. Nonetheless, we know from Proposition 5.4 that if $p\colon Y \to X$ is a crepant morphism of normal varieties over a field k of positive characteristic, then X is a splinter if and only if Y is a splinter. In the next paragraph, we will introduce the notion of $\mathcal{O}$-equivalence and will use it to circumvent going through small birational maps to improve upon Corollary 11.4 and show that the splinter property for Gorenstein varieties is invariant under K-equivalence; see Theorem 11.15.

11.2. **$\mathcal{O}$-equivalence.** Given a Noetherian scheme X, we denote by $\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_X)$ the derived category of complexes of $\mathcal{O}_X$-modules with coherent cohomology sheaves. Recall, e.g. from [Sta23, Tag 08E0], that the functor $\mathsf{D}^b(X) \to \mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_X)$ is fully faithful with essential image $\mathsf{D}^b_{\mathrm{Coh}}(\mathcal{O}_X)$. We use the formalism of the exceptional inverse image functor, as described in [Sta23, Tag 0A9Y] (under the name of *upper shriek functor*), for separated schemes of finite type over a fixed Noetherian base.

Note that the exceptional inverse image functor, when defined, does not preserve in general bounded complexes.

Definition 11.6 ($\mathcal{O}$-equivalence and strong $\mathcal{O}$-equivalence)**.** Let S be a Noetherian scheme and let X and Y be schemes of finite type and separated over S.

We say X and Y are *strongly* $\mathcal{O}$*-equivalent*, if there exists a scheme Z over S with proper birational S-morphisms $p\colon Z \to X$ and $q\colon Z \to Y$ such that $p^!\mathcal{O}_X \cong q^!\mathcal{O}_Y$ holds in $\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_Z)$.

We say X and Y are $\mathcal{O}$*-equivalent*, if there exists a scheme Z over S with proper birational S-morphisms $p\colon Z \to X$ and $q\colon Z \to Y$ and a proper surjective morphism $\mu\colon \tilde{Z} \to Z$ such that $\mu^! p^! \mathcal{O}_X \cong \mu^! q^! \mathcal{O}_Y$ holds in $\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_{\tilde{Z}})$.

Obviously, strong $\mathcal{O}$-equivalence implies $\mathcal{O}$-equivalence. As will become clear, $\mathcal{O}$-equivalence offers more generality and more flexibility than K-equivalence : in particular, it does not involve any ($\mathbb{Q}$-)Gorenstein assumption. For integral normal Gorenstein schemes, (strong) $\mathcal{O}$-equivalence and (strong) K-equivalence relate as follows :

Proposition 11.7. *Let X and Y be integral normal Gorenstein schemes of finite type and separated over an excellent Noetherian scheme S admitting a dualizing complex $\omega_S^\bullet$. Consider the following statements :*

(i) *X and Y are strongly K-equivalent.*
(ii) *X and Y are strongly $\mathcal{O}$-equivalent.*
(iii) *X and Y are K-equivalent.*
(iv) *X and Y are $\mathcal{O}$-equivalent.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv). If in addition X and Y are canonical, then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof. Let $p\colon Z \to X$ and $q\colon Z \to Y$ be birational morphisms of schemes of finite type and separated over a Noetherian scheme S admitting a dualizing complex. Recall from Lemma 2.2 that if X and Y are Gorenstein, then $p^!\mathcal{O}_X \cong q^!\mathcal{O}_Y$ if and only if $p^*\omega_X \cong q^*\omega_Y$. If X is reduced, then Z is generically reduced and hence [Sta23, Tag 0BXC] the normalization $Z^\nu \to Z$ is birational. We thus see that strong $\mathcal{O}$-equivalence coincides with strong K-equivalence, i.e., that (i) $\Leftrightarrow$ (ii).

The implication (ii) $\Rightarrow$ (iv) is clear and holds without assuming X and Y to be normal Gorenstein.

For (iii) $\Rightarrow$ (iv), consider the cyclic covering associated to the torsion line bundle $\mathcal{L} := p^*\omega_X \otimes q^*\omega_Y^{-1}$, i.e.,

$$\mu\colon \tilde{Z} := \operatorname{Spec}_Z\left(\bigoplus_{i=0}^{r-1} \mathcal{L}^i\right) \to Z,$$

where r is the torsion index of $\mathcal{L}$. As in the proof of Proposition 8.5, we have that $\mu^*\mathcal{L} = \mathcal{O}_{\tilde{Z}}$, i.e., $\mu^*p^*\omega_X \cong \mu^*q^*\omega_Y$. One concludes with Lemma 2.2 that $\mu^!p^!\mathcal{O}_X \cong \mu^!q^!\mathcal{O}_Y$.

For (iv) $\Rightarrow$ (iii), recall, e.g. from [Sta23, Tag 02U9], that for a proper morphism $\mu\colon \tilde{Z} \to Z$ and a line bundle $\mathcal{L}$ on Z we have

$$\mu_*(\mathrm{c}_1(\mu^*\mathcal{L}) \cap [\tilde{Z}]) = \mathrm{c}_1(\mathcal{L}) \cap \mu_*[\tilde{Z}] \quad \text{in } \mathrm{CH}_{\dim Z-1}(Z). \tag{11.8}$$

Assume now that X and Y are $\mathcal{O}$-equivalent and let $\mu\colon \tilde{Z} \to Z$, $p\colon Z \to X$ and $q\colon Z \to Y$ be as in Definition 11.6. Up to normalizing $\tilde{Z}$ and Z, we may and do assume that Z is normal. By Chow's Lemma [Sta23, Tag 02O2] and by taking hyperplane sections, we may assume that μ is generically finite. The condition $\mu^!p^!\mathcal{O}_X \cong \mu^!q^!\mathcal{O}_Y$ together with (11.8) then imply that the Weil divisor class associated to $p^*\omega_X \otimes q^*\omega_Y^{-1}$ is torsion, and hence by normality of Z that $p^*\omega_X \otimes q^*\omega_Y^{-1}$ is a torsion line bundle on Z, i.e. $p^*K_X \sim_{\mathbb{Q}} q^*K_Y$.

Finally if X and Y are canonical, then (i) $\Leftrightarrow$ (iii) by Remark 11.3. ☐

Moreover, under some regularity assumption, the cohomology of the structure sheaf is invariant under strong $\mathcal{O}$-equivalence :

Proposition 11.9. *Let X and Y be excellent regular schemes of finite type and separated over a Noetherian scheme S. If there exists an excellent regular scheme Z of finite type and separated over S with projective birational morphisms $p\colon Z \to X$ and $q\colon Z \to Y$ over S such that $p^!\mathcal{O}_X \cong q^!\mathcal{O}_Y$, then $H^i(X, \mathcal{O}_X) \cong H^i(Y, \mathcal{O}_Y)$ as $H^0(S, \mathcal{O}_S)$-modules for all $i \geq 0$.*

Proof. By [CR15, Thm. 1.1], both canonical maps $\mathcal{O}_X \to \mathrm{R}p_*\mathcal{O}_Z$ and $\mathcal{O}_Y \to \mathrm{R}q_*\mathcal{O}_Z$ are isomorphisms. Using the fact that the exceptional inverse image functor commutes with shifts, we obtain the chain of isomorphisms

$$\begin{aligned} H^i(Y,\mathcal{O}_Y) &= \mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_Y)}(\mathcal{O}_Y,\mathcal{O}_Y[i]) = \mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_Y)}(\mathrm{R}q_*\mathcal{O}_Z,\mathcal{O}_Y[i]) \\ &= \mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_Z)}(\mathcal{O}_Z, q^!\mathcal{O}_Y[i]) \cong \mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_Z)}(\mathcal{O}_Z, p^!\mathcal{O}_X[i]) \\ &= \mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_X)}(\mathrm{R}p_*\mathcal{O}_Z,\mathcal{O}_X[i]) = \mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_X)}(\mathcal{O}_X,\mathcal{O}_X[i]) = H^i(X,\mathcal{O}_X). \qquad \square \end{aligned}$$

Remark 11.10. Let X and Y be regular schemes of finite type and separated over a perfect field k. If X and Y are strongly $\mathcal{O}$-equivalent and if resolution of singularities holds for reduced schemes of dimension $\dim X$ over k, then by applying Chow's Lemma to Z as in Definition 11.6 and then resolving singularities, one may choose Z to be regular. Hence, provided resolution of singularities holds for reduced schemes of dimension $\dim X$ over k (which is the case if $\dim X \leq 3$ by [CP09]), Proposition 11.9 shows that if X and Y are strongly $\mathcal{O}$-equivalent, then $H^i(X,\mathcal{O}_X) \cong H^i(Y,\mathcal{O}_Y)$ for all $i \geq 0$.

11.3. ***D*-equivalence.** The following is classical.

Definition 11.11 (D-equivalence)**.** Two proper varieties X and Y over a field k are said to be *D-equivalent* (or *derived equivalent*) if there is a k-linear equivalence of categories $\mathsf{D}^b(X) \cong \mathsf{D}^b(Y)$ between their bounded derived categories of coherent sheaves.

Kawamata [Kaw02, Thm. 2.3(2)] showed that if X and Y are D-equivalent smooth projective varieties over an algebraically closed field of characteristic zero and if K_X or $-K_X$ is big, then X and Y are K-equivalent. We have the same result in the broader context of normal Gorenstein projective varieties over any field:

Proposition 11.12. *Let X and Y be normal Gorenstein projective varieties over a field k. Assume that K_X or $-K_X$ is big. If X and Y are D-equivalent, then X and Y are K-equivalent and, in particular, $\mathcal{O}$-equivalent.*

Proof. By [LO10, Cor. 9.17], a k-linear equivalence $\mathsf{D}^b(X) \cong \mathsf{D}^b(Y)$ is induced by a Fourier–Mukai transform with kernel $K \in \mathsf{D}^b(X \times_k Y)$. Denote by $\pi_X \colon X \times_k Y \to X$ and $\pi_Y \colon X \times_k Y \to Y$ the projections to X and Y. By [Her+09, Prop. 4.2], we have a natural isomorphism

$$\mathrm{R}\,\mathcal{H}om_{\mathcal{O}_{X\times_k Y}}(K, \pi_X^!\mathcal{O}_X) \cong \mathrm{R}\,\mathcal{H}om_{\mathcal{O}_{X\times_k Y}}(K, \pi_Y^!\mathcal{O}_Y). \tag{11.13}$$

(Note that the proof of [Her+09, Prop. 4.2] does not require that the base field be algebraically closed.) On the other hand, by base change [Sta23, Tag 0E9U] and since X and Y are Gorenstein, $\pi_X^!\mathcal{O}_X \cong \mathrm{L}\pi_Y^*\omega_Y^\bullet = \pi_Y^*\omega_Y[\dim Y]$ and $\pi_Y^!\mathcal{O}_Y \cong \mathrm{L}\pi_X^*\omega_X^\bullet = \pi_X^*\omega_X[\dim X]$. Thus, (11.13) is equivalent to

$$K^\vee \cong K^\vee \otimes \pi_X^*\omega_X \otimes \pi_Y^*\omega_Y^{-1}[\dim X - \dim Y] \quad \text{where } K^\vee := \mathrm{R}\,\mathcal{H}om_{\mathcal{O}_{X\times_k Y}}(K, \mathcal{O}_{X\times_k Y}).$$

As shown in [Her+09, Proof of Prop. 2.10], $K^\vee$ lies in $\mathsf{D}^b(X \times_k Y)$, and hence $\dim X = \dim Y$.

Let $\nu \colon Z^\nu \to Z$ be the normalization of an irreducible component Z of $\mathrm{Supp}(K^\vee)$ and set $p = \pi_X \circ \nu$ and $q = \pi_Y \circ \nu$. Then, there exists $i \in \mathbb{Z}$ such that $\nu^*\mathcal{H}^i(K^\vee)|_Z$ generically has positive rank $r > 0$. Arguing as in [Huy06, Lem. 6.9], we obtain

$$\mathcal{O}_{Z^\nu}(rp^*K_X) \cong \mathcal{O}_{Z^\nu}(rq^*K_Y).$$

Since K_X or $-K_X$ is big, arguing as in the proof of [Kaw02, Thm. 2.3(2)] (see also [Huy06, Prop. 6.19]) shows that there exists a component Z that dominates X and Y and is such that p and q are birational morphisms. This proves that X and Y are K-equivalent. By Proposition 11.7, X and Y are also $\mathcal{O}$-equivalent. $\square$

Remark 11.14. In the same way that D-equivalent smooth proper complex varieties are not necessarily strongly K-equivalent [Ueh04], D-equivalent smooth proper varieties in positive characteristic are not necessarily strongly $\mathcal{O}$-equivalent. Indeed, in [AB21], Addington and Bragg have produced D-equivalent smooth projective threefolds over $\overline{\mathbb{F}}_3$ with different Hodge numbers $h^{0,i}$ for $i = 1$ and 2. Such varieties are not $\mathcal{O}$-equivalent by Proposition 11.9 and Remark 11.10.

11.4. **Invariance of the splinter property under $\mathcal{O}$-equivalence.** Since a crepant morphism $p\colon Y \to X$ provides an $\mathcal{O}$-equivalence between Y and X, the following theorem generalizes statements (i) and (ii) of Proposition 5.4.

Theorem 11.15 (Theorem **(F)**, Derived splinters are stable under $\mathcal{O}$-equivalence)**.** *Let X and Y be integral schemes of finite type and separated over a Noetherian scheme S. If X and Y are $\mathcal{O}$-equivalent, then X is a derived splinter if and only if Y is a derived splinter.*

Proof. Suppose X is a derived splinter. Let $f\colon B \to Y$ be a proper surjective morphism. We have to show that $\eta_f\colon \mathcal{O}_Y \to \mathrm{R}f_*\mathcal{O}_B$ splits in $\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_Y)$. Consider the diagram

$$\begin{tikzcd}
 & \tilde{Z} \arrow[d, "\mu"] & & B' \arrow[ll, "f'"'] \arrow[dd, "q'"] \\
 & Z \arrow[dl, "p"'] \arrow[dr, "q"] & & \\
X & & Y & B \arrow[l, "f"']
\end{tikzcd}$$

where $\mu\colon \tilde{Z} \to Z$ is a proper surjective morphism over S, p and q are proper birational over S with $\mu^!p^!\mathcal{O}_X \cong \mu^!q^!\mathcal{O}_Y$, and where $B' := B \times_Y \tilde{Z}$. Let $s\colon \mathrm{R}p_*\mathrm{R}\mu_*\mathrm{R}f'_*\mathcal{O}_{B'} \to \mathcal{O}_X$ be a section of $\eta_{p\circ\mu\circ f'}\colon \mathcal{O}_X \to \mathrm{R}p_*\mathrm{R}\mu_*\mathrm{R}f'_*\mathcal{O}_{B'}$ and let $s'\colon \mathrm{R}q_*\mathrm{R}\mu_*\mathrm{R}f'_*\mathcal{O}_{B'} \to \mathcal{O}_Y$ be the image of s under the isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_X)}(\mathrm{R}p_*\mathrm{R}\mu_*\mathrm{R}f'_*\mathcal{O}_{B'}, \mathcal{O}_X) &= \mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_{\tilde{Z}})}(\mathrm{R}f'_*\mathcal{O}_{B'}, \mu^!p^!\mathcal{O}_X) \\ &\cong \mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_{\tilde{Z}})}(\mathrm{R}f'_*\mathcal{O}_{B'}, \mu^!q^!\mathcal{O}_Y) = \mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_Y)}(\mathrm{R}q_*\mathrm{R}\mu_*\mathrm{R}f'_*\mathcal{O}_{B'}, \mathcal{O}_Y).\end{aligned}$$

Choose dense open subsets $U \subseteq X$ and $V \subseteq Y$ such that $p^{-1}(U) = q^{-1}(V)$ and such that p and q restrict to isomorphisms over U and V, respectively. Then the composition

$$\mathcal{O}_Y \to \mathrm{R}q_*\mathrm{R}\mu_*\mathrm{R}f'_*\mathcal{O}_{B'} \xrightarrow{s'} \mathcal{O}_Y \tag{11.16}$$

restricted to V is isomorphic, via the isomorphism $p|_U \circ q|_V^{-1}\colon V \to U$, to

$$\mathcal{O}_X|_U \to \mathrm{R}p_*\mathrm{R}\mu_*\mathrm{R}f'_*\mathcal{O}_{B'}|_U \xrightarrow{s} \mathcal{O}_X|_U. \tag{11.17}$$

By choice of s, the composition in (11.17) is equal to the identity; hence, since Y is assumed to be integral, the composition in (11.16) sends the constant section $1_Y \in \mathcal{O}_Y(Y)$ to $1_Y \in \mathcal{O}_Y(Y)$. This shows that $\mathcal{O}_Y \to \mathrm{R}q_*\mathrm{R}\mu_*\mathrm{R}f'_*\mathcal{O}_{B'} = \mathrm{R}f_*\mathrm{R}q'_*\mathcal{O}_{B'}$ splits, and hence, that $\mathcal{O}_Y \to \mathrm{R}f_*\mathcal{O}_B$ splits. □

Combined with Proposition 11.12, we obtain a partial answer to the question, whether the splinter property for smooth projective schemes over a field is stable under derived equivalence:

Corollary 11.18 (Theorem **(D)**)**.** *Let X and Y be normal Gorenstein projective varieties over a field of positive characteristic. Assume that $-K_X$ is big. If X and Y are D-equivalent, then X is a splinter if and only if Y is a splinter.*

Proof. Recall from Bhatt [Bha12, Thm. 1.4] that a Noetherian scheme of positive characteristic is a splinter if and only if it is a derived splinter. By Proposition 11.12, X and Y are $\mathcal{O}$-equivalent, thus the statement follows from Theorem 11.15. □

Remark 11.19. According to Conjecture 3.12, the assumption that $-K_X$ is big in Corollary 11.18 is conjecturally superfluous.

11.5. **Invariance of global F-regularity under $\mathcal{O}$-equivalence and D-equivalence.**

Theorem 11.20 (Theorem **(G)**, Global F-regularity is stable under $\mathcal{O}$-equivalence)**.** *Let X and Y be quasi-projective varieties over an F-finite field k of positive characteristic. If X and Y are $\mathcal{O}$-equivalent, then X is normal globally F-regular if and only if Y is normal globally F-regular.*

Proof. Let $p\colon Z \to X$, $q\colon Z \to Y$ and $\mu\colon \tilde{Z} \to Z$ be as in Definition 11.6, and set $p' := p \circ \mu$ and $q' := q \circ \mu$. Assume that X is normal globally F-regular. By Proposition 3.10, X is a splinter. By [Bha12, Thm. 1.4], X is a derived splinter ; in particular the map $\mathcal{O}_X \to \mathrm{R}p_*\mathcal{O}_Z$ splits. Moreover, by Theorem 11.15, Y is also a derived splinter and hence is normal. Fix nonempty regular affine open subsets $U \subseteq X$ and $V \subseteq Y$ such that $p^{-1}(U) = q^{-1}(V)$ and such that $p|_U$ and $q|_V$ are isomorphisms. By [Sta23, Tag 0BCU], $D := X \setminus U$ and $E := Y \setminus V$ are divisors. Since Y is quasi-projective, any Weil divisor on Y is dominated by a Cartier divisor and we may thus further assume that E is Cartier. By Theorem 6.1 it suffices to show that there exists $e > 0$ such that $\mathcal{O}_Y \to F^e_*\mathcal{O}_Y(E)$ splits. Let $D' \geq D$ be a Cartier divisor on X. Since $q^*E = \sum_i a_iE_i$ for some $a_i \in \mathbb{Z}_{>0}$ with E_i the irreducible components of $\mathrm{Supp}(q^*E) \subseteq \mathrm{Supp}(p^*D')$, there exists $n \in \mathbb{Z}_{>0}$ such that $q^*E \leq np^*D'$. Let $U' := X \setminus D'$ and set $V' := q(p^{-1}(U'))$. Then U' and V' are regular affine open subsets such that $p|_{U'}$ and $q|_{V'}$ are isomorphisms.

Since X is globally F-regular, there exists an integer $e > 0$ such that $\sigma_{D'}\colon \mathcal{O}_X \to F^e_*\mathcal{O}_X(nD')$ splits. By the projection formula, the splitting of $\mathcal{O}_X \to \mathrm{R}p'_*\mathcal{O}_Z$ gives a splitting of $\mathrm{id}_{\mathrm{Coh}(X)} \to \mathrm{R}p'_*\mathrm{L}p'^*$. Thus, we obtain a splitting s of $\mathcal{O}_X \to F^e_*\mathcal{O}_X(nD') \to \mathrm{R}p'_*F^e_*\mathcal{O}_Z(np'^*D')$. Let s' be the image of s under the map

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_X)}(\mathrm{R}p'_*F^e_*\mathcal{O}_{\tilde{Z}}(np'^*D'), \mathcal{O}_X) &= \mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_{\tilde{Z}})}(F^e_*\mathcal{O}_{\tilde{Z}}(np'^*D'), p'^!\mathcal{O}_X)\\ &\to \mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_{\tilde{Z}})}(F^e_*\mathcal{O}_Z(q'^*E), q'^!\mathcal{O}_Y) = \mathrm{Hom}_{\mathsf{D}_{\mathrm{Coh}}(\mathcal{O}_Y)}(\mathrm{R}q'_*F^e_*\mathcal{O}_{\tilde{Z}}(q'^*E), \mathcal{O}_Y)\end{aligned}$$

induced by the inclusion $\mathcal{O}_{\tilde{Z}}(q'^*E) \hookrightarrow \mathcal{O}_{\tilde{Z}}(np'^*D')$ and the isomorphism $p'^!\mathcal{O}_X \cong q'^!\mathcal{O}_Y$. Consider the composition

$$\phi\colon \mathcal{O}_Y \to F^e_*\mathcal{O}_Y(E) \to F^e_*\mathrm{R}q'_*\mathrm{L}q'^*\mathcal{O}_Y(E) = \mathrm{R}q'_*F^e_*\mathcal{O}_{\tilde{Z}}(q'^*E) \xrightarrow{s'} \mathcal{O}_Y.$$

Under the isomorphism $p|_{U'} \circ q|_{V'}^{-1}$, $\phi|_{V'}$ corresponds to the composition

$$\mathcal{O}_X|_{U'} \to F^e_*\mathcal{O}_X(nD') \to \mathrm{R}p'_*F^e_*\mathcal{O}_{\tilde{Z}}(np'^*D')|_{U'} \xrightarrow{s} \mathcal{O}_X|_{U'}\,,$$

which is $\mathrm{id}_{\mathcal{O}_{U'}}$. Hence, since Y is integral, $\phi(1_Y) = 1_Y$ for the constant section $1_Y \in \mathcal{O}_Y(Y)$ and therefore $\phi = \mathrm{id}_{\mathcal{O}_Y}$. This shows that $\mathcal{O}_Y \to F^e_*\mathcal{O}_Y(E)$ splits. □

Remark 11.21 (The F-split property is stable under $\mathcal{O}$-equivalence). Assume that X and Y are normal quasi-projective varieties over an F-finite field k. If X and Y are $\mathcal{O}$-equivalent, then X is F-split if and only if Y is F-split. This follows indeed by considering, in the proof of Theorem 11.20, dense opens $U' = U \subseteq X$ and $V' = V \subseteq Y$ such that $p^{-1}(U) = q^{-1}(V)$ and such that $p|_U$ and $q|_V$ are isomorphisms, and by setting $D = 0$ and $E = 0$.

Corollary 11.22 (Theorem **(E)**, Global F-regularity is stable under D-equivalence)**.** *Let X and Y be normal Gorenstein projective varieties over an F-finite field of positive characteristic. If X and Y are D-equivalent, then X is globally F-regular if and only if Y is globally F-regular.*

Proof. Since normal globally F-regular projective varieties have big anticanonical class, this follows from Proposition 11.12 combined with Theorem 11.20. □

Remark 11.23 (The F-split property and D-equivalence). As asked by Zsolt Patakfalvi, we are unaware whether the F-split property is stable under D-equivalence. As a partial result, we mention that under the assumptions of Corollary 11.22, if X and Y are D-equivalent and if $-K_X$ is big, then X is F-split if and only if Y is F-split. For this, one argues as in the proof of Corollary 11.18 by using Remark 11.21.

Moreover, if X and Y are D-equivalent abelian varieties (resp. K3 surfaces, resp. strict Calabi–Yau threefolds), then X is F-split if and only if Y is F-split. This is classical in the case of abelian varieties and K3 surfaces, and is contained in [War14] in the case of strict Calabi–Yau threefolds with the additional assumption that these have vanishing first Betti number. Let us provide a proof. Under the assumptions above, we have an isomorphism of F-isocrystals $H^i_{\mathrm{crys}}(X) \cong H^i_{\mathrm{crys}}(Y)$ for $i = 1$ (resp. $i = 2$, resp. $i = 3$). For abelian varieties, this follows for instance from the more general fact that two D-equivalent smooth projective varieties have isogenous Albanese varieties [Hon18, Thm. B]. For K3 surfaces and strict Calabi–Yau threefolds, this follows from the general fact that the Mukai vector of the Fourier–Mukai kernel of a D-equivalence induces an isomorphism of F-isocrystals between the even-degree cohomologies and between the odd-degree cohomologies (see e.g. [Huy06, Prop. 5.33]) together with [Hon18, Thm. B] in the case of strict Calabi–Yau threefolds. In particular, if X and Y are D-equivalent abelian varieties (resp. K3 surfaces, resp. strict Calabi–Yau threefolds), then X and Y have the same height. One concludes with the fact that such varieties have height 1 if and only if they are F-split ; this is classical in the case of abelian varieties and is [GK03, Thm. 2.1] in the case of strict Calabi–Yau varieties.

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