Phantoms and Exceptional Collections on Rational Surfaces

Johannes Krah

A thesis submitted to the Faculty of Mathematics
at Bielefeld University in partial fulfillment of the requirements
for the degree of

Doctor of Mathematics

Faculty of Mathematics
Advisor: Prof. Dr. Charles Vial
Bielefeld University
Germany
Submitted: June 26, 2024
Revised: November 20, 2024

Acknowledgements

First and foremost, I want to express my gratitude to my advisor Charles Vial who initially sparked my interest in geometry and who guided me through my mathematical journey over the last years. From his proficiency of asking the right questions and his advice on how to write mathematics, this thesis benefited more than I could ever expect. Thank you, Charles, for answering all my questions and reading my manuscripts so many times!

I also wish to thank Henning Krause whose lectures shaped my mathematical foundations – including linear algebra in my first year in Bielefeld as well as derived categories in his advanced courses. Further, I thank Henning Krause and Alexander Kuznetsov for agreeing to serve as referees for this thesis.

I want to thank my colleagues and friends here in Bielefeld for the pleasant atmosphere – mathematically and non-mathematically as well as inside and outside the university.

On a more personal note, I want to thank my family for their support. Special thanks go to Elena. Thank you, Elena, for sharing all the aspects of my mathematical journey, proofreading all my writings, practicing all my talks, and constantly supporting the developments of this thesis.

The research was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 491392403 – TRR 358.

Printed on age-resistant paper °° ISO 9706.

Summary

This thesis is concerned with exceptional collections on smooth projective surfaces. Any rational surface over an algebraically closed field admits a full exceptional collection. We extend certain classification results regarding exceptional collections, previously known for del Pezzo surfaces, to the blow-up of the projective plane in 9 very general points. More generally, we obtain a classification result for numerically exceptional collections of maximal length on smooth projective surfaces S with $\chi(\mathcal{O}_S) = 1$ and $K_X^2 + \mathrm{rk}(\mathsf{K}_0^{\mathrm{num}}(X)) = 12$. In contrast to the case of 9 points, on the blow-up in 10 general points we construct an exceptional collection of maximal length which is not full. As a consequence, the orthogonal complement of this collection is a universal phantom category. This disproves a conjecture of Kuznetsov and a conjecture of Orlov.

Contents

Introduction

A theorem of Gabriel states that two algebraic varieties X and Y are isomorphic if and only if their categories of coherent sheaves $\mathrm{Coh}(X)$ and $\mathrm{Coh}(Y)$ are equivalent [Gab62, Ch. VI, Thm. 1]. Replacing coherent sheaves with bounded complexes of coherent sheaves and localizing at the class of quasi-isomorphisms, one obtains the derived category $\mathsf{D}^b(X) \coloneqq \mathsf{D}^b(\mathrm{Coh}(X))$. For algebraic varieties, the derived category yields a coarser invariant than the abelian category of coherent sheaves itself. For example, by the work of Mukai [Muk81, Thm. 2.2], the Poincaré bundle of an abelian variety A induces a derived equivalence to the dual abelian variety $\mathsf{D}^b(A) \cong \mathsf{D}^b(\hat{A})$. On the other hand, by Bondal–Orlov [BO01, Thm. 2.5], $\mathsf{D}^b(X)$ reflects the isomorphism class of X if X is smooth projective with ample or anti-ample canonical class K_X. By definition, $\mathsf{D}^b(X)$ contains the coherent cohomology of sheaves on X and suitably keeps track of relations among different complexes of sheaves. Moreover, if X is a smooth projective variety over the complex numbers $\mathbb{C}$, then $\mathsf{D}^b(X)$ captures the ungraded singular cohomology with rational coefficients $H^*(X, \mathbb{Q})$, see, e.g., [Huy06, Prop. 5.33]. Even finer, Orlov conjectured [Orl05, Conj. 1] that derived equivalent smooth projective varieties have isomorphic Chow motives with rational coefficient. In particular, evidence for this conjecture is provided by derived equivalences of irreducible holomorphic symplectic varieties: Taelman showed that if X and Y are derived equivalent irreducible holomorphic symplectic varieties, then $H^i(X, \mathbb{Q}) \cong H^i(Y, \mathbb{Q})$ as $\mathbb{Q}$-Hodge structures [Tae23, Thm. D]. These observations, as well as the triangulated structure $\mathsf{D}^b(X)$ is endowed with, turn the derived category into a reasonable as well as interesting invariant of algebraic varieties.

If X is a smooth projective variety over an algebraically closed field k, then dealing with the whole category $\mathsf{D}^b(X)$ as an invariant directly can be an ambitious goal. For that reason the notion of a so-called *semiorthogonal decomposition* proved to be a useful way of decomposing $\mathsf{D}^b(X)$ into smaller "pieces". The smallest possible piece in such a decomposition is a subcategory generated by an *exceptional object*, that is an object $E \in \mathsf{D}^b(X)$ such that $\mathrm{Hom}_{\mathsf{D}^b(X)}(E, E) = k$ and $\mathrm{Hom}_{\mathsf{D}^b(X)}(E, E[l]) = 0$ for all $l \in \mathbb{Z} \setminus \{0\}$. If E is an exceptional object, then $\langle E \rangle$, the full triangulated subcategory generated by E, is equivalent to the derived category of a single point, i.e., $\langle E \rangle \cong \mathsf{D}^b(\mathrm{Spec}\, k)$.

The "simplest" semiorthogonal decomposition of $\mathsf{D}^b(X)$, which one could hope for, consists only of exceptional objects and is called a *full exceptional collection.* That is, an ordered tuple $(E_1, \ldots, E_n) \subseteq \mathsf{D}^b(X)$ of exceptional objects such that $\mathrm{Hom}_{\mathsf{D}^b(X)}(E_i, E_j[l]) = 0$ for all $j < i$, $l \in \mathbb{Z}$ and such that $\langle E_1, \ldots, E_n \rangle = \mathsf{D}^b(X)$, i.e., $\mathsf{D}^b(X)$ is the smallest full triangulated subcategory which is closed under direct summands and contains $E_1, \ldots, E_n$. A full exceptional collection does not necessarily exist, but if it does, then it implies strong constraints on the geometry of the underlying variety. Most importantly, if $\mathsf{D}^b(X)$ admits a full exceptional collection $(E_1, \ldots, E_n)$, then the Grothendieck group is freely generated by the images of the E_i, i.e.,

$$\mathsf{K}_0(X) = \mathbb{Z}[E_1] \oplus \cdots \oplus \mathbb{Z}[E_n].$$

As a consequence, the Chow motive with rational coefficients of such a variety is a direct sum of Lefschetz motives [MT15, Thm. 1.1], [Via17, § 2]. For example, if X is defined over $\mathbb{C}$, this implies that the Hodge numbers $h^{p,q}(X)$ vanish whenever $p \neq q$.

Full exceptional collections were constructed for projective spaces [Bei78], Grassmannians and quadrics over $\mathbb{C}$ [Kap88], as well as certain other rational homogeneous varieties. Conjecturally any variety of the the form G/P, where G is a semi-simple algebraic group over $\mathbb{C}$ and $P \subseteq G$ a parabolic subgroup, admits a full exceptional collection [KP16, Conj. 1.1]. A folklore conjecture, attributed to Orlov, states that a variety with a full exceptional collection is rational.

This thesis is concerned with derived categories of rational surfaces. Over an algebraically closed field k, any rational surface X can be obtained from $\mathbb{P}^2_k$ or from a Hirzebruch surface Σ_d by a sequence of blow-ups in closed points. Hence, by Orlov's blow-up and projective bundle formulae for semiorthogonal decompositions [Orl92], any such X admits a full exceptional collection.

In Chapter 1 we investigate whether one can classify full exceptional collections on a given surface. One method to construct new collections from known ones are so-called *mutations*. A conjecture of Bondal–Polishchuk [BP93, Conj. 2.2] states that in a triangulated category any two full exceptional collections lie in the same orbit of the action by mutations and shifts. In this generality, the conjecture is known to be false [CHS23]. Nonetheless, the conjecture holds for $\mathsf{D}^b(X)$ if X is a del Pezzo surface [KO94]. The first result, Theorem 1.1.2, verifies the conjecture on the numerical level for triangulated categories $\mathsf{D}^b(X)$ where X is a smooth projective surface with $\chi(\mathcal{O}_X) = 1$ and $K_X^2 + \mathrm{rk}(\mathsf{K}_0^{\mathrm{num}}(X)) = 12$. More precisely, denoting $\chi(-,-)$ the Euler pairing, we consider mutations of *exceptional bases* in $\mathsf{K}_0^{\mathrm{num}}(X) := \mathsf{K}_0(X)/\ker\chi$, that is a basis $e_\bullet = (e_1, \dots, e_n)$ of $\mathsf{K}_0^{\mathrm{num}}(X)$ such that $\chi(e_i, e_i) = 1$ for all i and $\chi(e_i, e_j) = 0$ for all $j < i$.

Theorem (Theorem 1.1.2)**.** *Let X be a smooth projective surface over a field k with $\chi(\mathcal{O}_X) = 1$ and $K_X^2 + \mathrm{rk}(\mathsf{K}_0^{\mathrm{num}}(X)) = 12$. Let $e_\bullet$ and $f_\bullet$ be exceptional bases of $\mathsf{K}_0^{\mathrm{num}}(X)$.*

- (i) *There exists a $\mathbb{Z}$-linear automorphism $\phi\colon \mathsf{K}_0^{\mathrm{num}}(X) \to \mathsf{K}_0^{\mathrm{num}}(X)$ preserving the Euler pairing and the rank of elements such that $\phi(e_\bullet)$ can be transformed to $f_\bullet$ by a sequence of mutations and sign changes.*
- (ii) *If in addition $\mathrm{rk}\,\mathsf{K}_0^{\mathrm{num}}(X) \leq 12$, then $e_\bullet$ and $f_\bullet$ are related by a sequence of mutations and sign changes.*

Recall that, if k is algebraically closed, a del Pezzo surface is either isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$, or to a blow-up of $\mathbb{P}^2_k$ in up to 8 points with no 3 of them on a line, no 6 of them on a conic, and no 8 of them on a cubic having a node at one of them. In particular, a del Pezzo surface X satisfies $\chi(\mathcal{O}_X) = 1$ and $\mathrm{rk}\,\mathsf{K}_0^{\mathrm{num}}(X) \leq 11$. The second result, Theorem 1.1.3, concerns the particular case of $\mathbb{P}^2_{\mathbb{C}}$ blown up in 9 very general points. In this case $\mathrm{rk}\,\mathsf{K}_0^{\mathrm{num}}(X) = 12$.

Theorem (Theorem 1.1.3)**.** *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 9 very general points. Then*

- (i) *any numerically exceptional collection of maximal length consisting of line bundles is a full exceptional collection, and*
- (ii) *any two such collections are related by mutations and shifts.*

The above Theorem 1.1.3 (i) extends a result of Elagin–Lunts stating that on a del Pezzo surface any numerically exceptional collection of maximal length consisting of line bundles is a full exceptional collection [EL16].

The results obtained in Chapter 1 build on a detailed analysis of orthogonal transformations of the Picard groups. Namely, for X the blow-up of $\mathbb{P}^2_k$ in n points, we argue in

Lemmata 1.4.6 and 1.5.1 that

$$\mathrm{O}(\mathrm{Pic}(X))_{K_X} = \{f \in \mathrm{O}(\mathrm{Pic}(X)) \mid f(K_X) = K_X\} = \begin{cases} W_X & \text{if } n \leq 9, \\ W_X \times \langle \iota \rangle & \text{if } n = 10, \end{cases}$$

where $\mathrm{O}(\mathrm{Pic}(X))$ denotes the $\mathbb{Z}$-linear automorphisms of $\mathrm{Pic}(X)$ preserving the intersection form, $W_X \subseteq \mathrm{O}(\mathrm{Pic}(X))_{K_X}$ is the Weyl group associated to a certain root system in $\mathrm{Pic}(X)$, and ι is an involution of $\mathrm{Pic}(X)$ fixing the canonical class. A transformation in W_X can be expressed as a composition of simple reflections which correspond to *standard quadratic transformations* in the Cremona group of $\mathbb{P}^2_{\mathbb{C}}$. In contrast, the involution ι has no obvious geometric origin but still gives rise to a numerically exceptional collection. Intrigued by this observation, we study in Chapter 2 the numerically exceptional collection obtained by applying ι to a full exceptional collection consisting of line bundles. We show:

Theorem (Theorem 2.1.1)**.** *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 10 general points $p_1, \ldots, p_{10} \in \mathbb{P}^2_{\mathbb{C}}$. Denote by H the divisor class obtained by pulling back the class of a hyperplane in $\mathbb{P}^2_{\mathbb{C}}$ and denote by E_i the class of the exceptional divisor over the point p_i, $1 \leq i \leq 10$. Then*

$$\langle \mathcal{O}_X, \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F) \rangle \subseteq \mathsf{D}^b(X),$$

$$\textit{where} \quad D_i := \iota(E_i) = -6H + 2\sum_{j=1}^{10} E_j - E_i \quad \textit{and} \quad F := \iota(H) = -19H + 6\sum_{i=1}^{10} E_i,$$

is an exceptional collection of maximal length which is not full.

The above Theorem 2.1.1 disproves the following conjecture of Kuznetsov:

Conjecture ([Kuz14, Conj. 1.10])**.** *Let $\mathcal{T} = \langle E_1, \ldots, E_n \rangle$ be a triangulated category generated by an exceptional collection. Then any exceptional collection of length n in $\mathcal{T}$ is full.*

A nontrivial admissible subcategory $\mathcal{A} \subseteq \mathsf{D}^b(X)$ is a *phantom* if the Grothendieck group $\mathsf{K}_0(\mathcal{A})$ vanishes. Originally, there was hope that phantoms do not exist [Kuz09, Conj. 9.1]. Shortly after, the first examples of phantoms were constructed by Gorchinskiy–Orlov [GO13] and Böhning–Graf von Bothmer–Katzarkov–Sosna [BGKP15]. These examples are of two different types: The first example of Gorchinskiy–Orlov is constructed by taking the product of two surfaces which were known to admit *quasi-phantoms*, i.e., nontrivial admissible subcategories $\mathcal{A}$ such that $\mathsf{K}_0(\mathcal{A})$ is finitely generated and torsion. If one chooses two quasi-phantoms on suitable varieties such that the order of the torsion groups is coprime, then the product of the varieties carries a phantom category. The second example of Böhning–Graf von Bothmer–Katzarkov–Sosna considers a generic determinantal Barlow surface. Such a surface admits an exceptional collection of maximal length which is not full. Therefore, the left- or right-orthogonal complement of such a collection is a phantom category.

In contrast, it is known that del Pezzo surfaces do not admit phantom categories [Pir23, Thm. 6.35]. As in the case of the Barlow surface, it is a consequence of Theorem 2.1.1 that the right- or left-orthogonal complement of $\langle \mathcal{O}_X, \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F) \rangle$ is a phantom category and therefore provides an example of a phantom on a rational surface. As outlined in Section 2.1, the phantom category obtained this way can be identified with the homotopy category of the dg-category of perfect dg-modules over a smooth finite-dimensional dg-algebra. For that reason, Theorem 2.1.1 disproves the following conjecture of Orlov as well:

Conjecture ([Orl20, Conj. 3.7]). *There are no phantoms of the form* $\mathcal{P}erf-\mathcal{R}$*, where* $\mathcal{R}$ *is a smooth finite-dimensional dg-algebra and* $\mathcal{P}erf-\mathcal{R}$ *is the dg-category of perfect dg-modules over* $\mathcal{R}$*.*

Recent work of Borisov–Kemboi shows that the existence of the phantom in Theorem 2.1.1 depends on the *position* of the blown up points. It follows from [BK24, Thm. 1.1] that if X is the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in finitely many points in very general position on a smooth cubic curve, then $\mathsf{D}^b(X)$ contains no phantom. In order to obtain a deeper understanding why the phantom exists on blow-ups of 10 general points, it could be interesting to find a geometric interpretation of the symmetry ι used in Theorem 2.1.1. Note that ι as an involution of the Picard lattice exists regardless of the position of points. For that reason, it could be reasonable to expect that a geometric interpretation again incorporates the position of points.

Organization. Chapter 1 is based on the paper [Kra24b] which has been published in the *Mathematische Zeitschrift Volume 307, No. 4.* Chapter 2 is based on the paper [Kra24a] which has been published in *Inventiones mathematicae Volume 235, Issue 3.*

CHAPTER 1

Mutations of Numerically Exceptional Collections on Surfaces

Based on [Kra24b]

Summary. A conjecture of Bondal–Polishchuk states that, in particular for the bounded derived category of coherent sheaves on a smooth projective variety, the action of the braid group on full exceptional collections is transitive up to shifts. We show that the braid group acts transitively on the set of maximal numerically exceptional collections on rational surfaces up to isometries of the Picard lattice and twists with line bundles. Considering the blow-up of the projective plane in up to 9 very general points, these results lift to the derived category. More precisely, we prove that, under these assumptions, a maximal numerically exceptional collection consisting of line bundles is a full exceptional collection and any two of them are related by a sequence of mutations and shifts. The former extends a result of Elagin–Lunts and the latter a result of Kuleshov–Orlov, both concerning del Pezzo surfaces.

1.1. Introduction

Any smooth projective rational surface over an algebraically closed field admits a full exceptional collection by Orlov's projective bundle and blow-up formulae [Orl92], however a classification of exceptional collections on a given surface is widely open. To construct new exceptional collections from old ones, a key tool are so-called *mutations* of exceptional pairs, see Section 1.2.3; these give rise to an action of the braid group in n strands on the set of exceptional collections of length n on such a surface. Bondal and Polishchuk conjectured in more generality:

Conjecture 1.1.1 ([BP93, Conj. 2.2]). *Let $\mathcal{T}$ be a triangulated category which admits a full exceptional collection $\mathcal{T} = \langle E_1, \ldots, E_n \rangle$. Then any other full exceptional collection of $\mathcal{T}$ can be constructed from $\langle E_1, \ldots, E_n \rangle$ by a sequence of mutations and shifts.*

Recently, this conjecture was proven to be false [CHS23] and a counterexample is given by a Fukaya category of a certain smooth two-dimensional real manifold. To our knowledge, the conjecture still remains open for triangulated categories $\mathcal{T} = \mathsf{D}^b(\mathrm{Coh}(X))$, where X is a smooth projective variety.

This chapter is concerned with the question of classifying (numerically) exceptional collections on a given algebraic surface. Exceptional collections on rational surfaces have been previously studied in [HP11] and [Per18] via considering their associated toric surfaces. A classification of surfaces admitting a *numerically exceptional collection of maximal length* was carried out in [Via17]. Conjecture 1.1.1 was first verified in the cases $\mathcal{T} = \mathsf{D}^b(X)$, where X is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. The case where X is a del Pezzo surfaces is treated in [KO94]. In [Kul97], similar results for surfaces with basepoint-free anticanonical class were obtained. A full discussion of exceptional collections on the Hirzebruch surface Σ_2 was worked out in [IOU21] and Conjecture 1.1.1 was settled for $\mathsf{D}^b(\Sigma_2)$.

In the first part of the chapter, we consider the images of exceptional collections in $\mathsf{K}_0^{\mathrm{num}}(X)$ instead of the objects in the derived category itself. These so-called *numerically exceptional collections* on a surface X with $\chi(\mathcal{O}_X) = 1$ and $K_X^2 + \mathrm{rk}(\mathsf{K}_0^{\mathrm{num}}(X)) = 12$ have been previously investigated by Perling and Vial [Per18; Via17]. Their lattice-theoretic arguments have been reworked by Kuznetsov in the abstract setting of *surface-like pseudolattices*, introduced in [Kuz17]. Independently, a similar notion of a *surface-type Serre lattice* was developed in [dTVdB16]. In Section 1.2 we unify both formalisms in order to prove in Section 1.3 part (i) of the following

Theorem 1.1.2 (Theorem 1.3.1, Corollary 1.4.23)**.** *Let X be a smooth projective surface over a field k with $\chi(\mathcal{O}_X) = 1$ and $K_X^2 + \mathrm{rk}(\mathsf{K}_0^{\mathrm{num}}(X)) = 12$. Let $e_\bullet$ and $f_\bullet$ be exceptional bases of $\mathsf{K}_0^{\mathrm{num}}(X)$.*

- (i) *There exists a $\mathbb{Z}$-linear automorphism $\phi\colon \mathsf{K}_0^{\mathrm{num}}(X) \to \mathsf{K}_0^{\mathrm{num}}(X)$ preserving the Euler pairing and the rank of elements such that $\phi(e_\bullet)$ can be transformed to $f_\bullet$ by a sequence of mutations and sign changes.*
- (ii) *If in addition $\mathrm{rk}\,\mathsf{K}_0^{\mathrm{num}}(X) \leq 12$, then $e_\bullet$ and $f_\bullet$ are related by a sequence of mutations and sign changes.*

By definition, an exceptional basis of $\mathsf{K}_0^{\mathrm{num}}(X)$ is the class of a numerically exceptional collection of maximal length in $\mathsf{K}_0^{\mathrm{num}}(X)$, see Definition 1.2.2. Thus, we can reformulate Theorem 1.1.2 (i) as: Given two numerically exceptional collections $(E_1, \dots, E_n)$ and $(F_1, \dots, F_n)$ of maximal length on a surface X with $\chi(\mathcal{O}_X) = 1$ we can find a sequence of mutations and shifts σ such that $\chi(\sigma(E_i), \sigma(E_j)) = \chi(F_i, F_j)$ and $\mathrm{rk}\,\sigma(E_i) = \mathrm{rk}\,F_i$ holds for all $1 \leq i, j \leq n$.

Allowing automorphisms of $\mathsf{K}_0^{\mathrm{num}}(X)$ preserving χ in addition to mutations and shifts was classically considered in the case of $X = \mathbb{P}^2$, where full exceptional collections can be interpreted as solutions of the Markov equation, see, e.g., [GK04, § 7]. For lattices of higher rank this action was considered for instance in [Gor94].

To prove Theorem 1.1.2 (i) we can restrict to the case of X being either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blow-up of $\mathbb{P}^2$ in a finite number of points by using Vial's classification recalled in Theorem 1.2.13. Moreover, the group $\mathrm{Aut}(\mathsf{K}_0^{\mathrm{num}}(X)) = \mathrm{Aut}(\mathsf{K}_0^{\mathrm{num}}(X), \chi, \mathrm{rk})$ of isometries ϕ preserving the Euler pairing χ and the rank of elements fits into a short exact sequence

$$1 \to \mathrm{NS}(X) \to \mathrm{Aut}(\mathsf{K}_0^{\mathrm{num}}(X)) \to \mathsf{O}(\mathrm{NS}(X))_{K_X} \to 1,$$

where $\mathsf{O}(\mathrm{NS}(X))_{K_X} = \{f \in \mathsf{O}(\mathrm{NS}(X)) \mid f(K_X) = K_X\}$ is the stabilizer of the canonical class in the orthogonal group of $\mathsf{O}(\mathrm{NS}(X))$; see Lemma 1.2.11.

In Section 1.4 we address the question how to lift Theorem 1.1.2 (i) to $\mathsf{D}^b(X)$ and prove Theorem 1.1.2 (ii). The following two conditions are sufficient to deduce from Theorem 1.1.2 (i) that mutations and shifts act transitively on the set of full exceptional collections on X:

- (a) The action of an isometry $\phi\colon \mathsf{K}_0^{\mathrm{num}}(X) \to \mathsf{K}_0^{\mathrm{num}}(X)$ as in Theorem 1.1.2 (i) can be realized as a sequence of mutations and shifts.
- (b) Two full exceptional collections sharing the same class in $\mathsf{K}_0^{\mathrm{num}}(X)$ can be transformed into each other by a sequence of mutations and shifts.

If X is a del Pezzo surface, the arguments of [KO94] prove (b), see Lemma 1.4.11, and for the Hirzebruch surface Σ_2 the condition (b) is verified in [IOU21, § 6].

The main theorem of Elagin–Lunts in [EL16] states that any numerically exceptional collection consisting of line bundles on a del Pezzo surface is a full exceptional collection obtained from Orlov's blow-up formula applied to a minimal model. We extend this result to the blow-up X of 9 very general points in $\mathbb{P}^2_{\mathbb{C}}$.

Theorem 1.1.3 (Corollary 1.4.10, Theorem 1.4.18)**.** *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 9 very general points. Then*

(i) *any numerically exceptional collection of maximal length consisting of line bundles is a full exceptional collection, and*
(ii) *any two such collections are related by mutations and shifts.*

The position of the 9 points is discussed in Remark 1.4.4. Further, our results in Chapter 2 show that the statements of Theorem 1.1.3 do not hold for blow-ups of 10 or more points.

The proof of Theorem 1.1.3 (ii) is closely linked to the proof of Theorem 1.1.2 (ii). The key ingredient is the identification of the aforementioned group $\mathrm{O}(\mathrm{Pic}(X))_{K_X}$ with the Weyl group W_X of a root system embedded in $\mathrm{Pic}(X)$, see Lemma 1.4.6. Although this lattice-theoretic equality holds for the blow-up of up to 9 points regardless of their position, our argument relies on a result of Nagata [Nag60] which uses the actual geometry of X. The equality $\mathrm{O}(\mathrm{Pic}(X))_{K_X} = W_X$ then enables us to verify condition (a) for the blow-up in up to 9 very general points and thus we obtain Theorem 1.1.3 (ii) and Theorem 1.1.2 (ii).

In addition, our techniques provide a new proof of the fact that any two full exceptional collections on a del Pezzo surface are related by mutations and shifts; see Corollary 1.4.22. This result was proven in the first place by Kuleshov–Orlov in [KO94, Thm. 7.7].

Finally Section 1.5 discusses the lattice-theoretic behavior of the blow-up X of $\mathbb{P}^2$ in 10 points. In this case the Weyl group $W_X \subseteq \mathrm{O}(\mathrm{Pic}(X))_{K_X}$ has index two and $\mathrm{Pic}(X)$ admits an additional involution ι which fixes the canonical class K_X; see Lemma 1.5.1. While the action of W_X on exceptional collections of line bundles can be modeled by Cremona transformations of $\mathbb{P}^2$, the action of ι gives rise to an extraordinary numerically exceptional collection of line bundles. In Chapter 2 we show that the numerically exceptional collection obtained from ι is an exceptional collection of maximal length which is not full, provided the points are in general position. As a consequence, $\mathsf{D}^b(X)$ contains a phantom subcategory and the braid group action on exceptional collections of maximal length is not transitive. If one could verify condition (b) for exceptional collections of maximal length on X, the results of Chapter 2 would imply that the numerical bound in Theorem 1.1.2 (ii) is optimal, see Remark 1.5.4.

Conventions. In this chapter the term *surface* always refers to a smooth projective variety of dimension 2 over a field. The results in Section 1.3 are independent of the chosen base field, in Section 1.4 we exclusively work over the complex numbers. For a surface X, we write $\mathrm{NS}(X)$ for the Picard group modulo *numerical* equivalence. This coincides with the usual definition of the Néron–Severi group up to torsion if the base field is algebraically closed.

The term "n general points in $\mathbb{P}^2_{\mathbb{C}}$" means that there exists a nonempty Zariski open subset $U \subseteq (\mathbb{P}^2_{\mathbb{C}})^n$ such that for any $(p_1, \ldots, p_n) \in U$ [...] holds. The term "n very general points in $\mathbb{P}^2_{\mathbb{C}}$" means that there exist countably many nonempty Zariski open subset $U_i \subseteq (\mathbb{P}^2_{\mathbb{C}})^n$ such that for any $(p_1, \ldots, p_n) \in \bigcap_i U_i$ [...] holds.

Acknowledgements. We thank Pieter Belmans and Charles Vial for reading an earlier draft of [Kra24b] and we thank the anonymous referee of [Kra24b] for carefully reading our manuscript. Further, we thank Alexander Kuznetsov for his detailed comments on this chapter.

1.2. Numerically Exceptional Collections and Pseudolattices

We recall the necessary terminology of *surface-like pseudolattices* as it is presented in [Kuz17]. Independently, the notion of a *surface-type Serre lattice* was introduced in [dTVdB16]. After comparing both notions, we discuss the blow-up operation for pseudolattices in detail. Numerical blow-ups are explicitly mentioned in [dTVdB16] but were already used in [HP11; Via17; Kuz17] in a slightly different manner.

1.2.1. Exceptional collections. Let X be a smooth projective variety over a field k and let $\mathsf{D}^b(X) := \mathsf{D}^b(\mathrm{Coh}(X))$ be the bounded derived category of coherent sheaves on X. An object $E \in \mathsf{D}^b(X)$ is *exceptional* if $\mathrm{Hom}(E_i, E_i) = k$ and $\mathrm{Hom}(E_i, E_i[l]) = 0$ for all $l \neq 0$. A *full exceptional collection* in $\mathsf{D}^b(X)$ is a sequence of exceptional objects $(E_1, \ldots, E_n)$ such that $E_1, \ldots, E_n$ generate $\mathsf{D}^b(X)$ as a triangulated category and $\mathrm{Hom}(E_i, E_j[l]) = 0$ for all $l \in \mathbb{Z}$ whenever $i > j$. When considering only their images in the Grothendieck group $\mathsf{K}_0(X) := \mathsf{K}_0(\mathsf{D}^b(X))$ homomorphism spaces have to be exchanged with alternating sums over their dimensions. For this, let

$$\chi(E, F) := \sum_{j \in \mathbb{Z}} (-1)^j \dim_k \mathrm{Hom}(E, F[j])$$

be the *Euler pairing*. It gives rise to a bilinear form on $\mathsf{K}_0(X)$ and an object $E \in \mathsf{D}^b(X)$ is called *numerically exceptional* if $\chi(E, E) = 1$.

Definition 1.2.1. A *numerically exceptional collection* in $\mathsf{D}^b(X)$ is a sequence of numerically exceptional objects $(E_1, \ldots, E_n)$ such that $\chi(E_i, E_j) = 0$ whenever $i > j$. The sequence is said to be of *maximal length* if $[E_1], \ldots, [E_n] \in \mathsf{K}_0(X)$ generate $\mathsf{K}_0^{\mathrm{num}}(X)$ as a $\mathbb{Z}$-module or equivalently if $n = \mathrm{rk}\,\mathsf{K}_0^{\mathrm{num}}(X)$.

Here $\mathsf{K}_0^{\mathrm{num}}(X) := \mathsf{K}_0(X)/\ker\chi$ denotes the *numerical Grothendieck group*. Note that the left and right kernels of χ coincide thanks to Serre duality and note that $\mathsf{K}_0^{\mathrm{num}}(X)$ is torsion-free. Clearly χ defines a non-degenerate bilinear form on $\mathsf{K}_0^{\mathrm{num}}(X)$. Therefore studying numerically exceptional collections can be reduced to studying non-degenerate $\mathbb{Z}$-valued bilinear forms, which will be formalized in the notion of a *pseudolattice*.

1.2.2. Surface-like pseudolattices. We begin with recalling the notion of a pseudolattice in the sense of Kuznetsov.

Definition 1.2.2 ([Kuz17, Def. 2.1])**.** A *pseudolattice* is a finitely generated free abelian group G together with a non-degenerate bilinear form $\chi\colon G \otimes_{\mathbb{Z}} G \to \mathbb{Z}$. An *isometry* $\phi\colon (G, \chi_G) \to (H, \chi_H)$ between pseudolattices is a $\mathbb{Z}$-linear isomorphism which satisfies $\chi_G(v, w) = \chi_H(\phi(v), \phi(w))$ for all $v, w \in G$.

- The pseudolattice (G, χ) is *unimodular* if χ induces an isomorphism $G \to \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Z})$.
- Let $e_\bullet = (e_1, \ldots, e_n)$ be a basis of G, then $(\chi(e_i, e_j))_{i,j}$ is called the *Gram matrix* with respect to $e_\bullet$.
- An element $e \in G$ is called *exceptional* if $\chi(e, e) = 1$.
- An ordered basis $e_\bullet$ is called *exceptional basis* if the corresponding Gram matrix is upper unitriangular, i.e., $\chi(e_i, e_j) = 0$ whenever $i > j$ and $\chi(e_i, e_i) = 1$ for all i.
- A *Serre operator* is an isometry $S\colon G \to G$ satisfying $\chi(v, w) = \chi(w, S(v))$ for all $v, w \in G$.

Note that the lattice G is unimodular if and only if the Gram matrix has determinant ± 1. The Serre operator is unique, provided it exists, and if G is unimodular, it is given by $M^{-1}M^T$, where $M = (\chi(e_i, e_j))_{1 \le i,j \le n}$ is the Gram matrix of χ with respect to a chosen basis $(e_1, \ldots, e_n)$. In case we need to pass to rational coefficients, we use the notation $G_{\mathbb{Q}} := G \otimes_{\mathbb{Z}} \mathbb{Q}$ for a pseudolattice G (or more generally for any abelian group).

Definition 1.2.3 ([Kuz17, Def. 3.1])**.** A pseudolattice (G, χ) is *surface-like* if there exists a primitive element $\mathsf{p} \in G$ satisfying

(i) $\chi(\mathsf{p}, \mathsf{p}) = 0$,
(ii) $\chi(\mathsf{p}, v) = \chi(v, \mathsf{p})$ for all $v \in G$,
(iii) χ is symmetric on $\mathsf{p}^\perp \coloneqq \{v \in G \mid \chi(\mathsf{p}, v) = 0\}$.

Such an element p is called a *point-like element*.

The terminology is justified by the following geometric Example 1.2.5. First, recall that for a smooth projective surface S the Chern character induces an isomorphism

$$\mathrm{ch}\colon \mathsf{K}_0^{\mathrm{num}}(S)_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q} \oplus \mathrm{NS}(S)_{\mathbb{Q}} \oplus \mathbb{Q}, \tag{1.2.4}$$

see, e.g., [EL16, Lem. 2.1]. In particular, $\mathsf{K}_0^{\mathrm{num}}(S)$ has finite rank.

Example 1.2.5 (Pseudolattices from surfaces)**.** Let S be a smooth projective surface over a field k which admits a k-valued point $i\colon \{x\} \hookrightarrow S$. For example if S is rational, the existence of a k-valued point is guaranteed by the Lang–Nishimura Theorem. Let $G \coloneqq \mathsf{K}_0^{\mathrm{num}}(S)$ be the numerical Grothendieck group together with its Euler pairing. Then the class of the skyscraper sheaf $i_*k(x) = \mathcal{O}_x$ is a point-like element in G. An exceptional basis of G is the same as the image of a numerically exceptional collection of maximal length on S in $\mathsf{K}_0^{\mathrm{num}}(S)$.

Remark 1.2.6. More generally, any 0-cycle of degree 1 in $\mathrm{CH}_0(S)$ provides a point-like element by realizing it as a Chern character of a complex of skyscraper sheaves. However all the surfaces we consider are rational, for that reason we will only consider point-like elements as in Example 1.2.5.

From now on any surface-like pseudolattices $\mathsf{K}_0^{\mathrm{num}}(S)$, where S is a surface over k, is implicitly assumed to be endowed with the Euler pairing and a point-like element given by the class of a skyscraper sheaf of a k-valued point. Recall for later use that if $E, F \in \mathsf{D}^b(S)$ are objects with $e \coloneqq \mathrm{rk}\, E$ and $f \coloneqq \mathrm{rk}\, F$, then Riemann–Roch, see, e.g., [Per18, § 3.2], yields

$$\begin{aligned}\chi(E, F) = ef\chi(\mathcal{O}_X) + \frac{1}{2}\left(f\,\mathrm{c}_1(E)^2 + e\,\mathrm{c}_1(F)^2 - 2\,\mathrm{c}_1(E)\,\mathrm{c}_1(F)\right)\\ - \frac{1}{2}K_X(e\,\mathrm{c}_1(F) - f\,\mathrm{c}_1(E)) - (f\,\mathrm{c}_2(E) + e\,\mathrm{c}_2(F)).\end{aligned} \tag{1.2.7}$$

Given a surface-like pseudolattice G with point-like element p, we define the *rank function* with respect to p to be $\mathsf{r}(-) \coloneqq \chi(\mathsf{p}, -) = \chi(-, \mathsf{p})$. Then $\mathsf{p}^\perp = {}^\perp\mathsf{p} = \ker(\mathsf{r})$ and we obtain the analogue of the decomposition in (1.2.4).

Lemma 1.2.8 ([Kuz17, Lem. 3.10, Lem. 3.11])**.** *If G is a surface-like pseudolattice and p a point-like element, there is a complex*

$$\mathbb{Z} \xrightarrow{\mathsf{p}} G \xrightarrow{\mathsf{r}} \mathbb{Z}$$

with injective p and, if G is unimodular, surjective r. The middle cohomology of the above complex $\mathrm{NS}(G) \coloneqq \mathsf{p}^\perp/\mathsf{p}$ is a finitely generated free abelian group of rank $\mathrm{rk}(G) - 2$.

On $\mathrm{NS}(G)$ the pairing $-\chi$ induces a well-defined non-degenerate symmetric bilinear form q, called the intersection form, *which also will be denoted by the usual product $- \cdot -$.*

Lemma 1.2.9 ([Kuz17, Lem. 3.12])**.** *Let G be a surface-like pseudolattice with point-like element p and let $\lambda\colon \bigwedge^2 G \to \mathsf{p}^\perp$ be the alternating map sending $v \wedge w \mapsto \mathsf{r}(v)w - \mathsf{r}(w)v$. Then there is a unique element $K_G \in \mathrm{NS}(G)_{\mathbb{Q}}$, called* canonical class, *satisfying*

$$-\mathsf{q}(K_G, \lambda(v, w)) = \chi(v, w) - \chi(w, v) \tag{1.2.10}$$

for all $v, w \in G$*. If* G *is unimodular,* K_G *is integral, i.e.,* $K_G \in \mathrm{NS}(G)$*.*

The pair $(\mathrm{NS}(G), \mathsf{q})$ is called the *Néron–Severi lattice* and $\mathrm{NS}(G)$ the *Néron–Severi group*. One can check that for a surface S and pseudolattice $G = \mathsf{K}_0^{\mathrm{num}}(S)$ as in Example 1.2.5 all these definitions agree with the usual ones. For example, via Riemann–Roch (1.2.7) one computes $\chi(\mathcal{O}_x, \mathcal{F}) = \operatorname{rk} \mathcal{F}$ for any coherent sheaf $\mathcal{F}$ on S and $x \in S$ a k-valued point.

The following Lemma 1.2.11, which could not be found in the literature, will be important in the proof of Theorem 1.1.3. For that reason, we provide a proof here.

Lemma 1.2.11 (Self-isometries arise from orthogonal transformations)**.** *Let* G *be a surface-like pseudolattice of* $\operatorname{rk} G \geq 3$ *and let* $\mathrm{Aut}(G)$ *be the group of self-isometries* $\phi\colon G \to G$ *with* $\phi(\mathsf{p}) = \mathsf{p}$*. The map* $\Psi\colon \mathrm{Aut}(G) \to \mathrm{O}(\mathrm{NS}(G))$ *obtained by sending* $\phi \in \mathrm{Aut}(G)$ *to the induced orthogonal transformation of* $\mathrm{NS}(G)$ *defines a group homomorphism.*

Then, if G *is unimodular, the image of* Ψ *equals the stabilizer of the canonical class* $\mathrm{O}(\mathrm{NS}(G))_{K_G} = \{f \in \mathrm{O}(\mathrm{NS}(G)) \mid f(K_G) = K_G\}$*. Moreover, if* $G = \mathsf{K}_0^{\mathrm{num}}(X)$ *for some surface* X *with* $\chi(\mathcal{O}_X) = 1$ *as in Example 1.2.5, the kernel of* Ψ *can be identified with the subgroup of automorphisms given by twists with line bundles. In other words we obtain a short exact sequence*

$$1 \to \mathrm{NS}(X) \to \mathrm{Aut}(\mathsf{K}_0^{\mathrm{num}}(X)) \to \mathrm{O}(\mathrm{NS}(X))_{K_X} \to 1,$$

where $\mathrm{Aut}(\mathsf{K}_0^{\mathrm{num}}(X))$ *are the automorphisms preserving the point-like element* $[\mathcal{O}_x]$*.*

Proof. Since $\mathsf{r}(-) = \chi(\mathsf{p}, -)$, any $\phi\colon G \to G$ which preserves the point-like element p preserves the rank of elements. Hence it induces an orthogonal transformation of $\mathrm{NS}(G)$ which fixes the canonical class K_G. If G is unimodular, we can choose a rank 1 vector $v_0 \in G$. Since $\chi(v_0, v_0) = 1$, we have a splitting

$$G = {}^{\perp}v_0 \oplus \mathbb{Z}v_0 \quad \text{where} \quad {}^{\perp}v_0 := \{v \in G \mid \chi(v, v_0) = 0\}.$$

The inclusion ${}^{\perp}v_0 \cap \mathsf{p}^{\perp} \subseteq \mathsf{p}^{\perp}$ induces an isomorphism ${}^{\perp}v_0 \cap \mathsf{p}^{\perp} \xrightarrow{\sim} \mathrm{NS}(G)$ since any $D \in \mathrm{NS}(G) = \mathsf{p}^{\perp}/\mathsf{p}$ has a unique representative $d \in \mathsf{p}^{\perp}$ such that $\chi(d, v_0) = 0$. Any $\bar{\phi} \in \mathrm{O}(\mathrm{NS}(G))$ can be lifted to an automorphism of the sublattice

$$\mathrm{NS}(G) \oplus \mathbb{Z}\mathsf{p} = {}^{\perp}v_0 \cap \mathsf{p}^{\perp} \oplus \mathbb{Z}\mathsf{p} = \mathsf{p}^{\perp} \subseteq G$$

fixing p. If in addition $\bar{\phi}$ fixes K_G, then we can lift $\bar{\phi}$ to an automorphism ϕ of G fixing v_0 since

$$\chi(v_0, \mathsf{p}) = \chi(\mathsf{p}, v_0) = 1,$$
$$\chi(d, v_0) = 0, \text{ and } \chi(v_0, d) = -\mathsf{q}(K_G, d) \text{ for any } d \in {}^{\perp}v_0 \cap \mathsf{p}^{\perp} = \mathrm{NS}(G).$$

Now assume $G = \mathsf{K}_0^{\mathrm{num}}(X)$ for some surface X with $\chi(\mathcal{O}_X) = 1$ and let $\phi \in \mathrm{Aut}(G)$ be an isometry which is the identity on $\mathrm{NS}(G)$. The class $\phi([\mathcal{O}_X])$ is the class of a numerically exceptional object E of rank 1. Note that Riemann–Roch (1.2.7) implies that $\mathrm{c}_2(E) = 0$. Since

$$\mathrm{ch}(E) = \left(\operatorname{rk} E, \mathrm{c}_1(E), \frac{1}{2}(\mathrm{c}_1(E)^2 - 2\,\mathrm{c}_2(E))\right),$$

the condition $\mathrm{c}_2(E) = 0$ implies $[E] = [\mathcal{O}_X(\mathrm{c}_1(E))]$ in $\mathsf{K}_0^{\mathrm{num}}(X)$. Thus, twisting with $\mathcal{O}_X(\mathrm{c}_1(E))$ defines an isometry of G which maps $[\mathcal{O}_X]$ to $[E]$.

Let $F \in \mathsf{D}^b(X)$ be an object of rank 0. Then $\mathrm{ch}(F) = (0, \mathrm{c}_1(F), d)$, some $d \in \mathbb{Q}$. Multiplicativity of the Chern character gives

$$\mathrm{ch}(F(\mathrm{c}_1(E))) = (0, \mathrm{c}_1(F), d) \cdot (1, \mathrm{c}_1(E), d') = (0, \mathrm{c}_1(F), \mathrm{c}_1(F)\,\mathrm{c}_1(E) + d),$$

for some $d' \in \mathbb{Q}$. We observe that the first Chern class of F is invariant under twisting with a line bundle and also twisting with a line bundle does not change the point-like element defined by a skyscraper sheaf. Thus, $- \otimes \mathcal{O}_X(\mathrm{c}_1(E))$ is an isometry of G which fixes p, induces the identity on $\mathrm{NS}(G)$, and sends $[\mathcal{O}_X]$ to $[E]$. Let $v \in \mathsf{p}^\perp$, then $\phi(v) = v + n_v\mathsf{p}$ for some $n_v \in \mathbb{Z}$. On the other hand

$$\chi([\mathcal{O}_X], v) = \chi([E], \phi(v)) = \chi([E], v) + n_v\chi([E], \mathsf{p}) = \chi([E], v) + n_v,$$

i.e., $\phi(v) = v + \chi([\mathcal{O}_X] - \phi([\mathcal{O}_X]), v)\mathsf{p}$. Since ϕ was an arbitrary isometry in the kernel of Ψ, we must have $\phi(v) = v \otimes \mathcal{O}_X(\mathrm{c}_1(E))$ for every $v \in \mathsf{p}^\perp$. We conclude that $\phi = - \otimes \mathcal{O}_X(\mathrm{c}_1(E))$ since $G = \mathbb{Z}[\mathcal{O}_X] \oplus \mathsf{p}^\perp$. □

Definition 1.2.12 ([Kuz17, Def. 4.1, Lem. 4.2, Def. 4.3])**.** A surface-like pseudolattice G is called *geometric* if $(\mathrm{NS}(G), \mathsf{q})$ has signature $(1, \mathrm{rk}\, G - 3)$, the canonical class K_G is integral and K_G is *characteristic*, i.e., $\mathsf{q}(D, D) \equiv \mathsf{q}(K_G, D) (\mathrm{mod}\, 2)$ for all $D \in \mathrm{NS}(G)$. A surface-like pseudolattice G is *minimal* if it has no exceptional elements of rank zero. Equivalently $\mathrm{NS}(G)$ does not contain any (-1)-class.

It turns out that such geometric pseudolattices can be classified if we restrict to defect zero pseudolattices. Here the *defect* of G is the integer

$$\delta(G) \coloneqq K_G^2 + \mathrm{rk}(G) - 12.$$

If G is obtained as in Example 1.2.5 from a surface S with $\chi(\mathcal{O}_S) = 1$ and which admits a numerically exceptional collection of maximal length consisting of line bundles, one can show that $\delta(G) = 0$; see Remark 1.2.14 below. If S is a smooth projective surface over an algebraically closed field of characteristic such that $\mathcal{O}_S$ is exceptional, then, by [Kuz17, Lem. 5.5], $\delta(\mathsf{K}_0^{\mathrm{num}}(X)) = 0$.

Theorem 1.2.13 ([Via17, Thm. 3.1], [Kuz17, Thm. 5.12])**.** *Let G be a unimodular geometric pseudolattice of rank $n \geq 3$ and zero defect such that G represents 1 by a rank 1 vector, i.e., there exists $v \in G$ of rank 1 such that $\chi(v, v) = 1$. Then the following holds:*

- *$n = 3$ and $K_G = -3H$ for some $H \in \mathrm{NS}(G)$ if and only if G is isometric to $\mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^2)$;*
- *$n = 4$, $\mathrm{NS}(G)$ is even and $K_G = -2H$ for some $H \in \mathrm{NS}(G)$ if and only if G is isometric to $\mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^1 \times \mathbb{P}^1)$;*
- *$n \geq 4$, $\mathrm{NS}(G)$ is odd and K_G is primitive if and only if G is isometric to $\mathsf{K}_0^{\mathrm{num}}(X_{n-3})$.*

Here X_{n-3} is the blow-up of $\mathbb{P}^2$ in $n - 3$ points. Furthermore, G has an exceptional basis if and only if one of the three possibilities listed above is satisfied.

Remark 1.2.14. Let S be a smooth projective surface S over a field k with $\chi(\mathcal{O}_S) = 1$. Then, by [Via17, Thm. 3.1], S admits a numerically exceptional collection of maximal length consisting of line bundles if and only if $K_S^2 + \mathrm{rk}\, \mathrm{NS}(S) = 10$ and $(\mathrm{NS}(S), K_S)$ falls in one of the cases described in Theorem 1.2.13.

Let G be a surface-like pseudolattice with Serre operator S. Let $v \in G$, then

$$\chi(\mathsf{p}, (S - 1)(v)) = \chi(v, \mathsf{p}) - \chi(\mathsf{p}, v) = 0.$$

Furthermore, [Kuz17, Lem. 3.14] shows that $S - 1$ maps $\mathsf{p}^\perp$ to $\mathbb{Z}\mathsf{p}$. Thus, we obtain a decreasing filtration

$$F^3G = 0 \subseteq F^2G = \mathbb{Z}\mathsf{p} \subseteq F^1G = \mathsf{p}^\perp \subseteq F^0G = G$$

such that $S - 1$ maps F^iG to $F^{i+1}G$. If G is unimodular, the rank map induces an isomorphism $\mathsf{r}\colon G/\mathsf{p}^\perp \to \mathbb{Z}$, thus the above filtration defines a so-called *codimension filtration*.

Definition 1.2.15 ([dTVdB16, Def. 5.1.1])**.** Let G be a pseudolattice with Serre operator S and let $V \coloneqq G_{\mathbb{Q}}$. A *codimension filtration* on V is a filtration

$$0 = F^3V \subseteq F^2V \subseteq F^1V \subseteq F^0V = V$$

such that $(S-1)(F^iV) \subseteq F^{i+1}V$, $\dim F^0V/F^1V = \dim F^2 = 1$ and $\chi(F^1V, F^2V) = 0$.

Conversely, any codimension filtration gives rise to a point-like element by choosing a generator of $F^2G = F^2V \cap G$. This yields a 1:1-correspondence

$$\{\text{codimension filtrations } F^\bullet \text{ on } G\} \leftrightarrow \{\text{point-like elements } \mathsf{p}\}/\{\pm 1\}.$$

We will refer to both of them, a point-like element and a codimension filtration, as a *surface-like structure* on the pseudolattice G. In Example 1.2.5 the codimension filtration coincides with the topological codimension filtration, as discussed in [Kuz17, Ex. 3.5].

1.2.3. Mutations. Given $e \in G$ we define the *left mutation* L_e and its *right mutation* R_e as

$$\mathsf{L}_e(v) \coloneqq v - \chi(e,v)e$$
$$\mathsf{R}_e(v) \coloneqq v - \chi(v,e)e$$

for all $v \in G$. Note that the left and right mutation define mutually inverse isomorphisms of the orthogonal complements of e

$$^{\perp}e \underset{\mathsf{R}_e}{\overset{\mathsf{L}_e}{\rightleftarrows}} e^{\perp}.$$

Given an exceptional basis $e_\bullet = (e_1, \ldots, e_n)$ of G we define

$$\mathsf{L}_{i,i+1}(e_\bullet) \coloneqq (e_1, \ldots, e_{i-1}, \mathsf{L}_{e_i}(e_{i+1}), e_i, e_{i+2}, \ldots, e_n),$$
$$\mathsf{R}_{i,i+1}(e_\bullet) \coloneqq (e_1, \ldots, e_{i-1}, e_{i+1}, \mathsf{R}_{e_{i+1}}(e_i), e_{i+2}, \ldots, e_n).$$

The sequences are again exceptional bases and the above operations are mutually inverse. By construction, these mutations match the known mutations of exceptional collections if $G = \mathsf{K}_0^{\mathrm{num}}(S)$ as in Example 1.2.5. Indeed, if S is a surface and $E \in \mathsf{D}^b(S)$ an exceptional object, the *left mutation* L_E and *right mutation* R_E are defined as

$$\mathsf{L}_E(F) \coloneqq \mathrm{Cone}\left(E \otimes \mathrm{RHom}(E,F) \xrightarrow{\mathrm{ev}} F\right) \text{ and}$$
$$\mathsf{R}_E(F) \coloneqq \mathrm{Cone}\left(F \xrightarrow{\mathrm{ev}^\vee} E \otimes \mathrm{RHom}(F,E)^\vee\right)[-1]$$

for any object $F \in \mathsf{D}^b(S)$. Note that by construction the diagram

$$\begin{array}{ccc} \mathsf{D}^b(S) & \longrightarrow & \mathsf{K}_0^{\mathrm{num}}(S) \\ \downarrow{\scriptstyle \mathsf{M}_E} & & \downarrow{\scriptstyle \mathsf{M}_{[E]}} \\ \mathsf{D}^b(S) & \longrightarrow & \mathsf{K}_0^{\mathrm{num}}(S) \end{array}$$

commutes, where $\mathsf{M}_E = \mathsf{L}_E$ or $\mathsf{M}_E = \mathsf{R}_E$.

Moreover, if $\mathsf{D}^b(S) = \langle E_1, \ldots, E_n \rangle = \langle E_\bullet \rangle$ is a full exceptional collection, the sequences

$$\mathsf{L}_{i,i+1}(E_\bullet) \coloneqq (E_1, \ldots, E_{i-1}, \mathsf{L}_{E_i}(E_{i+1}), E_i, E_{i+2}, \ldots, E_n),$$
$$\mathsf{R}_{i,i+1}(E_\bullet) \coloneqq (E_1, \ldots, E_{i-1}, E_{i+1}, \mathsf{R}_{E_{i+1}}(E_i), E_{i+2}, \ldots, E_n).$$

are again full exceptional collections. Already on the level of $\mathsf{D}^b(S)$ the operations $\mathsf{L}_{i,i+1}$ and $\mathsf{R}_{i,i+1}$ give rise to an action of the braid group $\mathfrak{B}_n$, see, e.g., [BP93, Prop. 2.1]. Together with $\mathbb{Z}^n$ acting by shifts, this yields an action of the semidirect product $\mathbb{Z}^n \rtimes \mathfrak{B}_n$ on the set of full exceptional collections, where the homomorphism $\mathfrak{B}_n \to \operatorname{Aut}(\mathbb{Z}^n)$ is the composition of the canonical map $\mathfrak{B}_n \to \mathfrak{S}_n$ and the action of $\mathfrak{S}_n$ on $\mathbb{Z}^n$ by permutations. If two exceptional bases lie in the same orbit of the $\mathbb{Z}^n \rtimes \mathfrak{B}_n$-action, we say the exceptional collections are *related by mutations up to shifts.*

On the level of $\mathsf{K}_0^{\mathrm{num}}(S)$ shifts result in sign changes. More generally, if G is a pseudolattice of rank n with exceptional basis, then $\{\pm 1\}^n \rtimes \mathfrak{B}_n$ acts on the set of exceptional bases, where $\{\pm 1\}^n$ acts by changing signs of basis elements. Moreover, this action commutes with the action of isometries $\phi\colon G \to G$. If two exceptional bases lie in the same orbit of $\{\pm 1\}^n \rtimes \mathfrak{B}_n$, we say the exceptional bases are *related by mutations up to signs.* In this chapter, we will only consider pseudolattices with surface-like structure. If we write that two exceptional bases $e_\bullet, f_\bullet$ are *related by mutations up to signs and isometry* we mean that there exists an isometry $\phi\colon G \to G$ which preserves the point-like element $\phi(\mathsf{p}) = \mathsf{p}$ and $\phi(e_\bullet)$ and $f_\bullet$ are related by mutations up to signs.

Let G be a surface-like pseudolattice with exceptional basis. We will frequently mutate to norm-minimal bases, where the *norm* of an exceptional basis $e_\bullet = (e_1, \dots, e_n)$ is the number $\sum_i \mathsf{r}(e_i)^2$. We say an exceptional basis is *norm-minimal* if there is no exceptional basis related by mutations and sign changes with smaller norm. Recall that due to the work of Perling, norm-minimal exceptional bases can be understood via *Perling's algorithm*:

Theorem 1.2.16 ([Kuz17, Thm. 5.9], cf. [Per18, Cor. 9.12, Cor. 10.7])**.** *Let G be a geometric surface-like pseudolattice. Any exceptional basis in G can be transformed by mutations and sign changes into a norm-minimal exceptional basis consisting of* 3 *or* 4 *elements of rank* 1 *and all other elements of rank* 0*.*

1.2.4. Blow-up and blow-down. We recall the classical blow-up and blow-down construction for surface-like pseudolattices and give a detailed discussion of [dTVdB16, § 5] as we make use of these observations in Section 1.3. Let G be a unimodular surface-like pseudolattice with point-like element p. We denote the induced codimension filtration by $F^\bullet G$. Let $e_\bullet = (e_1, \dots, e_n)$ be a basis of G and let M be the Gram matrix of the pairing χ with respect to this basis. Choosing an element $z \in F^2 G = \mathbb{Z}\mathsf{p}$, we construct the *numerical blow-up* of G at z as follows: We extend the lattice G by adding a formal element f, i.e., we consider the free abelian group $\operatorname{Bl}_z G \coloneqq \mathbb{Z}f \oplus G$. The pairing χ_{new} on $\operatorname{Bl}_z G$ is defined via

$$\begin{aligned} \chi_{\mathrm{new}}|_{G\otimes G} &\coloneqq \chi, \\ \chi_{\mathrm{new}}(g, f) &\coloneqq 0 \text{ for all } g \in G, \\ \chi_{\mathrm{new}}(f, f) &\coloneqq 1, \\ \text{and } \chi_{\mathrm{new}}(f, g) &\coloneqq \chi(z, g) \text{ for all } g \in G. \end{aligned}$$

In abuse of notation we write χ also for the pairing on $\operatorname{Bl}_z G$. As outlined below, this definition matches the geometric situation of a blow-up. The Gram matrix with respect to the basis $(f, e_1, \dots, e_n)$ is of the form

$$\left(\begin{array}{c|ccc} 1 & \chi(z, e_1) & \cdots & \chi(z, e_n) \\ \hline 0 & & & \\ \vdots & & M & \\ 0 & & & \end{array}\right).$$

Note that $\operatorname{Bl}_z G$ is again unimodular and surface-like with point-like element $\mathsf{p} \in G \subseteq \mathbb{Z}f \oplus G$. The latter follows from writing $z = n\mathsf{p}$ for some $n \in \mathbb{Z}$ which shows $\chi(\mathsf{p}, f) = 0 = \chi(z, \mathsf{p}) = \chi(f, \mathsf{p})$. The orthogonal complement of p in $\operatorname{Bl}_z G$ is $F^1 \operatorname{Bl}_z G = F^1 G \oplus \mathbb{Z}f$ and χ is symmetric on $F^1 G \oplus \mathbb{Z}f$ as it is symmetric on both summands and $\chi(F^1 G, f) = 0 =$

$n\chi(\mathsf{p}, F^1G) = \chi(f, F^1G)$. In particular, $F^2 \operatorname{Bl}_z G = \mathbb{Z}\mathsf{p} = F^2G$. Therefore the point-like element p does not change under blow-up; this allows us to blow up the same element multiple times. Note that the image of f in

$$\operatorname{NS}(\operatorname{Bl}_z G) = \operatorname{NS}(G) \overset{\perp}{\oplus} \mathbb{Z}f$$

defines an element of self-intersection -1. It is the analogue of a (-1)-curve and can be blown down, but in contrast to the geometric setting, we cannot detect whether a divisor of self-intersection -1 is an actual curve or not.

Again we compare the construction to the geometric one (cf. Example 1.2.5). Let S be a smooth projective surface and let $\tilde{S}$ be the blow-up at a point $p \in S$ with exceptional divisor E:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\pi} & S \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle i} \\ E & \xrightarrow{\psi} & \{p\}. \end{array}$$

For $\mathcal{F} \in \mathsf{D}^b(S)$ a Riemann–Roch computation shows

$$\chi(j_*\mathcal{O}_E(-1)[1], \mathrm{L}\pi^*F) = -\chi(j_*\mathcal{O}_E(-1), \mathrm{L}\pi^*F) = \operatorname{rk}(\mathcal{F}) = \chi_S(\mathcal{O}_p, \mathcal{F}),$$

see [Per18, Ex. 4.1]. Finally, Orlov's blow-up formula yields a semiorthogonal decomposition

$$\mathsf{D}^b(\tilde{S}) = \langle j_*(\psi^*\mathsf{D}^b(\{p\}) \otimes \mathcal{O}_E(-1)), \mathrm{L}\pi^*\mathsf{D}^b(S)\rangle = \langle j_*\mathcal{O}_E(-1), \mathrm{L}\pi^*\mathsf{D}^b(S)\rangle$$

which coincides with the numerical blow-up construction with $f = [j_*\mathcal{O}_E(-1)[1]]$.

The inverse operation on a unimodular surface-like pseudolattice G is the *blow-down* or *contraction*. Let $f \in G$ be a rank zero vector such that $\mathsf{q}(f,f) = -\chi(f,f) = -1$. Then the contraction of f is the lattice $G_f := {}^\perp f = \{v \in G \mid \chi(v,f) = 0\} \subseteq G$ with pairing $\chi|_{^\perp f \otimes {}^\perp f}$. The pseudolattice G_f is again surface-like with point-like element p and unimodular; see [Kuz17, Lem. 5.1]. If G is geometric, so is G_f. In the following we prove a slightly modified version of [dTVdB16, Lem. 5.1], which will be a key tool towards establishing Theorem 1.3.1.

Proposition 1.2.17. *Let G be a unimodular surface-like pseudolattice and $f \in G$ a rank zero vector of self-intersection -1. Denote by S the Serre operator of G, then $z := (S-1)(f) \in F^2G_f$ defines an element such that $\operatorname{Bl}_z G_f = G$.*

Proof. Since $f \in F^1G = \mathsf{p}^\perp$, we know that $(S-1)(f) \in F^2G = \mathbb{Z}\mathsf{p}$ is a multiple of p and lies in G_f. Let $H := \operatorname{Bl}_z(G_f) = \mathbb{Z}g \oplus G_f$ be the blow-up of G_f at z. Then the pairing on H extends the pairing of G_f with the property that g is an element of rank zero and of self-intersection -1. Consider the morphism $G \to H$ sending $f \mapsto g$ and $v \mapsto v$ for all $v \in G_f$. We verify that this is an isometry: Obviously $\chi(g,g) = 1 = \chi(f,f)$ and $\chi(v,f) = 0 = \chi(v,g)$ for $v \in G_f$. Let $v \in G_f$, then

$$\chi(g,v) = \chi(z,v) = \chi(v,z) = \chi(v, S(f)) - \chi(v,f) = \chi(f,v),$$

where we have used that z is a multiple of p and $\chi(v,f) = 0$ for all $v \in G_f$. □

Clearly $(\operatorname{Bl}_z G)_f = G$, thus blow-up and blow-down are mutually inverse.

Remark 1.2.18. Comparing the blow-down construction described above to the construction in [Kuz17, § 5], one observes that the contraction of an exceptional element of rank zero can also be defined as the right orthogonal $f^\perp$ since left and right orthogonal complements are isomorphic.

We end this section by recalling formulae for the defect of the contraction.

Lemma 1.2.19 ([Kuz17, Lem. 5.8])**.** *Let G be a surface-like pseudolattice and $f \in G$ an exceptional element of rank zero. Then the defect of G equals*

$$\delta(G) = \delta(G_f) + (1 - \mathsf{q}(K_G, f)^2).$$

If G is geometric, then $\delta(G) \leq \delta(G_f)$ with equality if and only if $\mathsf{q}(K_G, f) = \pm 1$.

In the same manner a formula for the degree of the blow-up was obtained in [dTVdB16, Lem. 5.2.1]. The *degree* of a unimodular surface-like pseudolattice G is $\deg(G) = K_G^2$ and is related to the defect by the formula

$$\deg(G) = 12 + \delta(G) - \mathrm{rk}(G). \tag{1.2.20}$$

Lemma 1.2.21 ([dTVdB16, Lem. 5.2.1])**.** *Let G be a unimodular surface-like pseudolattice and let $\sigma \in G$ be an element such that its image $\bar{\sigma}$ generates $\mathrm{Bl}_z G / F^1 \mathrm{Bl}_z G \cong \mathbb{Z}$. Then* $\deg \mathrm{Bl}_z G = \deg G - \chi(\sigma, z)^2$.

Lemma 1.2.21 above requires a justification in our context, as it is possibly not clear that the canonical class in the sense of [dTVdB16] coincides with the one in Lemma 1.2.9.

Proof of Lemma 1.2.21. The image ω of $(S-1)(\sigma)$ in $\mathrm{NS}(G)$ is the canonical class of G in the sense of [dTVdB16, Def. 3.5.1] and in [dTVdB16, Lem. 5.2.1] the statement is shown for $\deg G := \mathsf{q}(\omega, \omega)$. Therefore it is enough to show:

Claim. Let G be a unimodular surface-like pseudolattice, $\sigma \in G/F^1 G$ a generator and $\omega = (S-1)(\sigma) \in \mathrm{NS}(G)$. Then ω satisfies (1.2.10) up to sign, i.e.,

$$\pm \mathsf{q}(\omega, \lambda(v, w)) = \chi(v, w) - \chi(w, v)$$

for all $v, w \in G$ and λ as in Lemma 1.2.9.

Proof of the Claim. Since G is unimodular, the rank map induces an isomorphism $\mathsf{r}\colon G/F^1 G \to \mathbb{Z}$. Let $\sigma \in G$ be a vector such that $\bar{\sigma}$ generates $G/F^1 G$. Up to possibly replacing σ by $-\sigma$ we can write any $v \in G$ as $\mathsf{r}(v)\sigma + \tau(\sigma)$ with $\tau(\sigma) \in F^1 G$. Let $\omega = (S-1)(\sigma)$ be the canonical class defined by σ and let $d(v) := \mathsf{q}(\tau(v), \omega)$ for all $v \in G$. By [dTVdB16, Prop. 3.6.2] the equality

$$\chi(v, w) - \chi(w, v) = \det \begin{pmatrix} d(v) & d(w) \\ \mathsf{r}(v) & \mathsf{r}(w) \end{pmatrix} = \mathsf{r}(w)\mathsf{q}(\tau(v), \omega) - \mathsf{r}(v)\mathsf{q}(\tau(w), \omega) \tag{1.2.22}$$

holds for all $v, w \in G$. Let λ be the alternating form as in Lemma 1.2.9. Then

$$\begin{aligned} -\mathsf{q}(\omega, \lambda(v, w)) &= -\mathsf{q}(\mathsf{r}(v)(\mathsf{r}(w)\sigma + \tau(w)) - \mathsf{r}(w)(\mathsf{r}(v)\sigma + \tau(v)), \omega) \\ &= -\mathsf{q}(\mathsf{r}(v)\mathsf{r}(w)\sigma + \mathsf{r}(v)\tau(w) - \mathsf{r}(w)\mathsf{r}(v)\sigma - \mathsf{r}(w)\tau(v), \omega) \\ &= \mathsf{q}(\mathsf{r}(w)\tau(v) - \mathsf{r}(v)\tau(w), \omega) \end{aligned}$$

combined with (1.2.22) proves the claim. □

1.3. Proof of Theorem 1.1.2 (i)

Throughout this section let X_k be the blow-up of $\mathbb{P}^2$ in k distinct points and let $G_k := \mathsf{K}_0^{\mathrm{num}}(X_k)$ be the pseudolattice obtained from X_k. Using Vial's classification, see Theorem 1.2.13 and Remark 1.2.14, we can rephrase Theorem 1.1.2 (i) as follows:

Theorem 1.3.1. *Let $e_\bullet$ and $f_\bullet$ be two exceptional bases of G_k or of $\mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^1 \times \mathbb{P}^1)$. Then there exists an isometry $\phi\colon G \to G$ preserving the surface-like structure, i.e., $\phi(\mathsf{p}) = \mathsf{p}$, such that $\phi(e_\bullet)$ and $f_\bullet$ are related by mutations up to signs.*

In particular, note that for the Hirzebruch surfaces Σ_d, by Theorem 1.2.13, we have that

$$\mathsf{K}_0^{\mathrm{num}}(\Sigma_d) \cong \begin{cases} \mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^1 \times \mathbb{P}^1) & \text{if } d \text{ is even,} \\ \mathsf{K}_0^{\mathrm{num}}(\mathrm{Bl}_p\, \mathbb{P}^2) & \text{if } d \text{ is odd.} \end{cases}$$

In preparation for the proof of Theorem 1.3.1 we compute an explicit form of the pseudolattices G_k. The surface $\mathbb{P}^2$ admits a full exceptional sequence consisting of line bundles, namely the Beilinson sequence $\mathsf{D}^b(\mathbb{P}^2) = \langle \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)\rangle$. This yields an exceptional basis of the numerical Grothendieck group $G_0 := \mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^2) = \mathbb{Z}[\mathcal{O}_{\mathbb{P}^2}] \oplus \mathbb{Z}[\mathcal{O}_{\mathbb{P}^2}(1)] \oplus \mathbb{Z}[\mathcal{O}_{\mathbb{P}^2}(2)]$ with Gram matrix

$$M_0 := \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Lemma 1.3.2. *Let $e_\bullet = (e_1, e_2, e_3)$ be an exceptional basis of G_0 with Gram matrix M_0, then a point-like element is given by $\mathsf{p} := e_3 - 2e_2 + e_1$.*

Proof. For a closed point $i\colon \{x\} \hookrightarrow \mathbb{P}^2$ the skyscraper-sheaf $i_*k(x) = \mathcal{O}_x$ admits a Koszul resolution

$$[0 \to \mathcal{O}_{\mathbb{P}^2}(-2) = \bigwedge^2 \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2} \to 0] \cong \mathcal{O}_x.$$

Thus, we obtain after twisting by $\mathcal{O}_{\mathbb{P}^2}(2)$

$$[\mathcal{O}_x] = [\mathcal{O}_{\mathbb{P}^2}(2)] - 2[\mathcal{O}_{\mathbb{P}^2}(1)] + [\mathcal{O}_{\mathbb{P}^2}] = e_3 - 2e_2 + e_1 \in \mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^2).$$ □

Remark 1.3.3. The point-like element can also be computed directly from the pseudolattice using the explicit description of [dTVdB16, Lem. 3.3.2]. Namely if $V := G_{\mathbb{Q}}$, then $F^2V = \mathrm{Im}(S-1)^2$ and $F^2V = \mathbb{Q}\mathsf{p}$. Thus p spans the line $\mathrm{Im}(S-1)^2$ over $\mathbb{Q}$ and is primitive. In the case of $\mathbb{P}^2$ one computes $S = M_0^{-1}M_0^T$ and

$$(S-1)^2 = \begin{pmatrix} 9 & 9 & 9 \\ -18 & -18 & -18 \\ 9 & 9 & 9 \end{pmatrix}.$$

Hence, $\mathsf{p} = \pm(1, -2, 1)$.

By the blow-up formula we compute Gram matrices M_k of the pseudolattices G_k, namely

$$M_k := \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & 1 & \cdots & 0 & 1 & 1 & 1 \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 3 & 6 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 3 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{c|ccc} & 1 & 1 & 1 \\ \mathrm{id}_{k\times k} & \vdots & \vdots & \vdots \\ & 1 & 1 & 1 \\ \hline 0 & & M_0 & \end{array}\right),$$

since $\chi(\mathsf{p}, e_i) = 1$ for $i = 1, 2, 3$. Denote by $b_1, \dots, b_k, e_1, e_2, e_3$ the exceptional basis corresponding to this Gram matrix. The elements b_i are all orthogonal to p, so of rank zero and the corresponding images in $\mathrm{NS}(G)$ have self-intersection -1. Here, we have numerically blown up the point p in order to obtain only positive signs in the Gram matrix. We first verify Theorem 1.3.1 in the minimal cases:

Proposition 1.3.4 ([Kuz17, Cor. 4.24])**.** *Any two exceptional bases in $\mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^2)$ are related by mutations up to sign and isometry.*

Proof. By [Kuz17, Cor. 4.24] norm-minimal exceptional bases of $\mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^2)$ correspond to the Beilinson sequence $\langle \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)\rangle$. Therefore any two exceptional bases are related by mutations up to sign and isometry. □

Proposition 1.3.5. *Any two exceptional bases in* $\mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^1 \times \mathbb{P}^1)$ *are related by mutations up to sign and isometry.*

Proof. If G admits a norm-minimal basis consisting of objects of nonzero rank, G is isometric to $\mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^1 \times \mathbb{P}^1)$ and the norm-minimal basis corresponds to one of the full exceptional collections

$$\mathsf{D}^b(\mathbb{P}^1 \times \mathbb{P}^1) = \langle \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(c,1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(c+1,1)\rangle$$

for some $c \in \mathbb{Z}$; see [Kuz17, Cor. 4.26]. The corresponding Gram matrix is

$$D_c := \begin{pmatrix} 1 & 2 & 2c+2 & 2c+4 \\ 0 & 1 & 2c & 2c+2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

see [Kuz17, Ex. 3.7]. Now we mutate the third and fourth basis vector and compute the corresponding Gram matrices:

$$\mathsf{L}_{3,4}(b_1,\dots,b_4) = (b_1, b_2, -2b_3 + b_4, b_3), \qquad \begin{pmatrix} 1 & 2 & -(2(c-1)+2) & 2(c-1)+4 \\ 0 & 1 & -2(c-1) & 2(c-1)+2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathsf{R}_{3,4}(b_1,\dots,b_4) = (b_1, b_2, b_4, b_3 - 2b_4), \qquad \begin{pmatrix} 1 & 2 & 2(c+1)+2 & -(2(c+1)+4) \\ 0 & 1 & 2(c+1) & -(2(c+1)+2) \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Multiplying $-2b_3 + b_4$ by -1 in the first case and $b_3 - 2b_4$ in the second case, we observe that all bases corresponding with Gram matrices D_c are related by mutations up to sign and isometry. □

For later use in the proof of Theorem 1.3.1, we also treat the surfaces X_1 and X_2 by hand.

Proposition 1.3.6. *Any two exceptional bases in* $\mathsf{K}_0^{\mathrm{num}}(X_1)$ *are related by mutations up to sign and isometry.*

Proof. Since $G_1 := \mathsf{K}_0^{\mathrm{num}}(X_1)$ is not isometric to $\mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^1 \times \mathbb{P}^1)$, [Kuz17, Cor. 4.27] shows that an exceptional basis of G_1 with all ranks nonzero is not norm-minimal. Thus, by Perling's algorithm Theorem 1.2.16, a norm-minimal basis has the form $e_1, \dots, e_4$, where e_1 is of rank zero and e_2, e_3 and e_4 are of rank one. By Vial's classification Theorem 1.2.13, the contraction $(G_1)_{e_1}$ is isomorphic to $G_0 = \mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^2)$ with norm-minimal exceptional basis e_2, e_3, e_4. Since blow-up and blow-down are mutually inverse, e_1 results from blowing up a point $z = n\mathsf{p} \in \mathbb{Z}\mathsf{p}$. Observe that $\delta(G_1) = \delta((G_1)_{e_1}) = 0$, so by (1.2.20) the degree has to decrease by 1 from G_1 to $(G_1)_{e_1}$. Thus by Lemma 1.2.21 $n = \pm 1$ and $\chi(\sigma, z) = \pm 1$. Possibly after changing the sign of e_1, the Gram matrix with respect to $e_1, \dots, e_4$ is M_1. □

The surface X_2 can be obtained from blowing up $\mathbb{P}^2$ in 2 points or from blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ in 1 point. So a priori, there could potentially be two different types of norm-minimal exceptional bases. We compute that this is not the case.

Proposition 1.3.7. *Let X_2 be the blow-up of $\mathbb{P}^2$ in 2 points and let $G_2 := \mathsf{K}_0^{\mathrm{num}}(X_2)$. Then any two exceptional bases are related by mutations up to sign and isometry. In particular, any norm-minimal exceptional basis is of norm 3.*

Proof. We show that any exceptional basis can be mutated to an exceptional basis with Gram matrix M_2. Let $e_\bullet$ be an exceptional basis. Again with Perling's algorithm we mutate $e_\bullet$ to a norm-minimal basis $a_1, \dots, a_l, b_1, \dots, b_m$ with a_i of rank zero and b_i of rank one. Now $m \in \{3, 4\}$, since the (iterated) contraction of the rank zero elements yields a minimal geometric surface-like pseudolattice, which admits an exceptional basis; that implies it is isometric to $\mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^2)$ or to $\mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^1 \times \mathbb{P}^1)$. Assume for contradiction $m = 4$. Then the contraction $(G_2)_{a_1}$ has Gram matrix

$$D_c := \begin{pmatrix} 1 & 2 & 2c+2 & 2c+4 \\ 0 & 1 & 2c & 2c+2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with respect to $b_1, \dots, b_4$, see [Kuz17, Ex. 3.7]. As mutations in the contraction $(G_2)_{a_1}$ lift to mutations of G_2 which leave the contracted vectors invariant, we can assume that $c = 0$ by Proposition 1.3.5. Moreover, as in the proof of Proposition 1.3.6, a_1 is obtained by blowing up $\mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^1 \times \mathbb{P}^1)$ in $\pm\mathsf{p}$. After possibly changing the sign of a_1, we can assume a_1 results from blowing up p. Now we want to find a sequence of mutations, which reduces the norm of $(a_1, b_1, \dots, b_4)$. We compute:

$$\begin{aligned}
&(a_1, b_1, b_2, b_3, b_4) \\
\xrightarrow{\mathsf{L}_{1,2}}\ &(-a_1 + b_1, a_1, b_2, b_3, b_4) \\
\xrightarrow{\mathsf{R}_{2,3}}\ &(-a_1 + b_1, b_2, a_1 - b_2, b_3, b_4) \\
\xrightarrow{\mathsf{R}_{3,4}}\ &(-a_1 + b_1, b_2, b_3, a_1 - b_2 - b_3, b_4) \\
\xrightarrow{\mathsf{L}_{1,2}}\ &(a_1 - b_1 + b_2, -a_1 + b_1, b_3, a_1 - b_2 - b_3, b_4) \\
\xrightarrow{\mathsf{L}_{2,3}}\ &(a_1 - b_1 + b_2, a_1 - b_1 + b_3, -a_1 + b_1, a_1 - b_2 - b_3, b_4) \\
\xrightarrow{\mathsf{R}_{4,5}}\ &(a_1 - b_1 + b_2, a_1 - b_1 + b_3, -a_1 + b_1, b_4, a_1 - b_2 - b_3 + 3b_4).
\end{aligned} \tag{1.3.8}$$

Since the rank map is additive one easily computes that the last basis is of rank $(0, 0, 1, 1, 1)$. But this contradicts the assumption that $(a_1, b_1, \dots, b_4)$ was norm-minimal. Thus $m = 3$ and the exceptional basis a_1, a_2, b_1, b_2, b_3 results from blowing up $\mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^2)$ in 2 points $n_1\mathsf{p}$ and $n_2\mathsf{p}$. After possibly changing signs, we can assume $n_1, n_2 \geq 0$. The fact that G_2 and $(G_2)_{a_1,a_2} = G_0$ have defect zero implies that also $(G_2)_{a_1}$ has defect zero, since contraction only increases the defect by Lemma 1.2.19. Therefore the degree has to increase by 1 in each contraction and we have $n_1 = n_2 = 1$ by Lemma 1.2.21. Hence, the Gram matrix with respect to to a_1, a_2, b_1, b_2, b_3 is M_2. □

Remark 1.3.9. One can further compute the Gram matrix with respect to $(a_1 - b_1 + b_2, a_1 - b_1 + b_3, -a_1 + b_1, b_4, a_1 - b_2 - b_3 + 3b_4)$ as

$$\begin{pmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Further, the collection above corresponds to the full exceptional collection

$$\mathsf{D}^b(\mathrm{Bl}_p\,\mathbb{P}^1\times\mathbb{P}^1) = \langle \mathcal{O}_{\tilde{l}_1}(-1)[1], \mathcal{O}_{\tilde{l}_2}(-1)[1], \mathcal{O}_{\mathrm{Bl}_p\,\mathbb{P}^1\times\mathbb{P}^1}(E), \mathcal{O}_{\mathrm{Bl}_p\,\mathbb{P}^1\times\mathbb{P}^1}(H_1+H_2), \mathcal{O}_{\mathrm{Bl}_p\,\mathbb{P}^1\times\mathbb{P}^1}(2H_1+2H_2-E)\rangle$$

up to shifts, where $p=(p_1,p_2)\in\mathbb{P}^1\times\mathbb{P}^1$, H_1 and H_2 are the pullbacks of hyperplane classes on the corresponding factors, E is the exceptional divisor over p, H_1-E is the class of the strict transform $\tilde{l}_1$ of the line $\{p_1\}\times\mathbb{P}^1$, and H_2-E is the class of the strict transform $\tilde{l}_2$ of the line $\mathbb{P}^1\times\{p_2\}$. Both computations will not be used subsequently.

Proof of Theorem 1.3.1. We show the following statement: Given any exceptional basis $a_\bullet$ of G_k we can find another exceptional basis related by mutations and sign changes to $a_\bullet$ such that the Gram matrix is of the form M_k and the first k basis elements have rank zero and the last 3 have rank one. The unimodularity then ensures that the involved isometry preserves the surface-like structure given by p, since the isometry respects the rank function. As we have treated the cases $k\le 2$ by hand, we can assume $k>2$. Given any exceptional basis of G_k we can mutate the basis to a norm-minimal basis

$$(a_1,\dots,a_l,b_1,\dots,b_m)$$

where the elements a_i are of rank zero and the b_i are of rank one and m is equal to 3 or 4; see Theorem 1.2.16. Contracting the rank zero objects a_i, we obtain a minimal unimodular geometric surface-like pseudolattice $(G_k)_{a_1,\dots,a_l}$ admitting an exceptional basis. Thus the defect of $(G_k)_{a_1,\dots,a_l}$ is zero by [Kuz17, Cor. 5.7]. Contraction of geometric pseudolattices only increases the defect, cf. Lemma 1.2.19, thus all intermediate pseudolattices $(G_k)_{a_1,\dots,a_i}$ are unimodular geometric surface-like pseudolattices with defect zero and admit an exceptional basis. This implies that they are isometric to blow-ups of $\mathsf{K}_0^{\mathrm{num}}(\mathbb{P}^2)$ as long as $k-i\ge 2$ by Theorem 1.2.13. Choosing i such that $k-i=2$, the pseudolattice $(G_k)_{a_1,\dots,a_i}$ is isometric to the blow-up of 2 points in $(G_k)_{a_1,\dots,a_i}$. By Proposition 1.3.7 any two exceptional bases of $(G_k)_{a_1,\dots,a_i}$ are related by mutations up to sign and we can mutate the exceptional basis to a basis of norm 3. Now mutations in the contraction lift to mutations of G_k, which leave the contracted vectors invariant. Hence, $m=3$ and $l=k$. In particular, $(G_k)_{a_1,\dots,a_k}$ is isometric to G_0 and we may assume that b_1,b_2,b_3 have Gram matrix M_0.

As seen in Section 1.2.4, blowing up and contracting are mutually inverse operations. Thus the basis $a_1,\dots,a_k,b_1,b_2,b_3$ is a basis obtained from blowing up G_0 in k points. The point-like element of G_0 is unique up to sign, as discussed in [Kuz17, Ex. 3.5], hence we can assume $\mathsf{p}=b_3-2b_2+b_1$. In each intermediate step $(G_k)_{a_1,\dots,a_{i+1}}$ is obtained from $(G_k)_{a_1,\dots,a_i}$ by blowing up a point $n_{i+1}\mathsf{p}$ with $n_{i+1}\in\mathbb{Z}$. As each $(G_k)_{a_1,\dots,a_j}$ has defect zero we deduce $n_j=\pm1$ for all j. Indeed, by (1.2.20) the degree has to decrease by -1 in each step and Lemma 1.2.21 yields

$$\deg((G_k)_{a_1,\dots,a_i}) = \deg((G_k)_{a_1,\dots,a_{i+1}}) - \chi(\sigma, n_{i+1}\mathsf{p})^2 = \deg((G_k)_{a_1,\dots,a_{i+1}}) - n_{i+1}^2.$$

Up to possibly changing signs, we can arrange $\chi(a_i,b_j)=1$ for all i,j. Thus the Gram matrix has the desired form. □

1.4. Blow-up of 9 Points

Full exceptional collections on del Pezzo surfaces were studied in [KO94] and in [EL16]. In [EXZ21] and [IOU21] similar results for *weak del Pezzo surfaces*, i.e., surfaces with nef and big anticanonical divisor, were obtained. In this section, we expand the class of examples by considering the blow-up of $\mathbb{P}^2$ in 9 very general points. In this situation, we can assume that there is a unique cubic curve in $\mathbb{P}^2$ passing through each of the 9 points with multiplicity 1. Then the divisor class of the strict transform of this cubic coincides with the anticanonical divisor $-K_X = 3H - \sum_{i=1}^{9} E_i$ of the blow-up X. Here H is the pullback of a hyperplane class in $\mathbb{P}^2$ and E_i is the exceptional divisor corresponding to the blow-up of the point p_i. Therefore, $-K_X$ is nef but not big as $(-K_X)^2 = 0$, so X is not a weak del Pezzo surface. Additionally, $-K_X$ is not basepoint-free and for that reason the techniques developed in [Kul97] cannot be applied. In this section we exclusively work over the field of complex numbers.

1.4.1. Toric systems and numerically exceptional collections. We recall the necessary terminology of toric systems as introduced by Hille–Perling in [HP11, §§ 2-3].

Definition 1.4.1. Let X be a smooth projective surface. A sequence of divisors $A_1, \ldots, A_n$ on X is a *toric system* if $n \geq 3$ and one has $A_i \cdot A_{i+1} = 1 = A_1 \cdot A_n$ for all $1 \leq i \leq n-1$, $A_i \cdot A_j = 0$ for $|i-j| > 1$ except $\{i,j\} = \{1,n\}$, and $A_1 + \cdots + A_n \sim_{\mathrm{lin}} -K_X$.

If $\chi(\mathcal{O}_X) = 1$ and $n = \operatorname{rk} \mathsf{K}_0(X)$, we have a 1:1-correspondence between toric systems on X and numerically exceptional collections consisting of line bundles of length n up to twists with line bundles:

$$\{\text{toric systems } (A_1, \ldots, A_n)\} / \sim_{\mathrm{lin}} \leftrightarrow \left\{ \begin{matrix} \text{numerically exceptional collections} \\ \text{of line bundles } (\mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_n)) \end{matrix} \right\} / \mathrm{Pic}(X).$$

Although not stated in this way, the correspondence can be deduced from [Via17, Prop. 2.1] (a system of divisors as in [Via17, Prop. 2.1 (iii)] defines a toric system by [Via17, Prop. A.3] where K_X is the *special characteristic element*). A toric system $(A_1, \ldots, A_n)$ and a choice of a divisor D_1 defines a numerically exceptional collection $(\mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_n))$ given by

$$D_{i+1} \coloneqq D_1 + A_1 + \cdots + A_i.$$

Conversely, any numerically exceptional collection of line bundles $(\mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_n))$ gives rise to a toric system via

$$A_i \coloneqq \begin{cases} D_{i+1} - D_i & \text{for } 1 \leq i \leq n-1, \\ D_1 - K_X - D_n & \text{for } i = n. \end{cases}$$

A toric system is called *exceptional* if the corresponding collection of line bundles is exceptional. Equivalently, each divisor $A_i + \cdots + A_j$ $(1 \leq i \leq j \leq n-1)$ is *left-orthogonal* (a divisor D is called left-orthogonal if $h^i(-D) = 0$ for all i). Moreover, $(A_1, \ldots, A_n)$ is an (exceptional) toric system if and only if $(A_2, \ldots, A_n, A_1)$ is an (exceptional) toric system.

Orlov's blow-up formula for full exceptional collections can be transferred to toric systems via so-called *augmentations*; see [HP11, § 5] and [EL16, § 2.6]: If X' is a surface with toric system $A'_1, \ldots, A'_n$ and $p\colon X \to X'$ the blow-up of X in a closed point $p \in X'$ with exceptional divisor $E \subseteq X$, denote by $A_i \coloneqq p^* A'_i$ the pullback of the divisors. We obtain a toric system on X, namely

$$E, A_1 - E, A_2, \ldots, A_{n-1}, A_n - E.$$

This toric system and all its cyclic shifts are called *augmentations*. Conversely, a blow-down operation for toric systems can be defined.

Proposition 1.4.2 ([EL16, Prop. 3.3])**.** *Let $A_1, \ldots, A_n$ be a toric system on a surface X such that there exists an index $1 \leq m \leq n$ with A_m a (-1)-curve in X. Let $p\colon X \to X'$ be the blow-down of A_m. Then $A_1, \ldots, A_n$ is an augmentation of a toric system $A'_1, \ldots, A'_{n-1}$ on X'.*

An essential observation for the proof of Theorem 1.1.3 is that [EL16, Lem. 3.4] generalizes to the blow-up of $\mathbb{P}^2$ in 9 points. Recall that a (-1)-curve in a surface S is a smooth rational curve in S of self-intersection -1.

Lemma 1.4.3. *Let X be the blow-up of $\mathbb{P}^2$ in 9 points in very general position. Then a divisor D is the class of a (-1)-curve if and only if $D^2 = -1$ and $\chi(D) = 1$.*

Proof. First of all, Riemann–Roch yields $\chi(D) = 1 + \frac{1}{2}(D(D - K_X))$. Thus, if D is the class of a (-1)-curve, then $D^2 = -1$ and $\chi(D) = 1$. For the converse direction let D be a divisor with $D^2 = -1$ and $\chi(D) = 1$. As $D^2 = -1$ we have $-K_X D = 1$. Now $\chi(D) = h^0(D) - h^1(D) + h^2(D)$ and $h^2(D) = h^0(K_X - D)$ by Serre duality. The intersection $-K_X(K_X - D) = K_X D = -1$ implies that $K_X - D$ is not effective, since $-K_X$ is nef. Therefore $h^2(D) = h^0(K_X - D) = 0$ and in order for $\chi(D) = 1$ to be fulfilled, D must have at least one nontrivial global section, i.e., D must be effective. We write $D = \sum_i k_i C_i$, where the C_i are pairwise distinct integral curves in X and the k_i are positive integers. From the equation $1 = -K_X D = \sum_i k_i(-K_X)C_i$ and the nefness of $-K_X$ we derive that among the curves C_i there is one C_0 occurring with coefficient 1 and satisfying $-K_X C_0 = 1$. All other C_i lie in $K_X^\perp$. Note that by [Fer05, Prop. 2.3] any integral curve with negative self-intersection is a (-1)-curve. Therefore no curve in $K_X^\perp$ can have negative self-intersection, as for (-1)-curves the intersection with the canonical class is nonzero. Hence, in order to achieve $-1 = D^2$ we must have $C_0^2 = -1$, $C_i^2 = 0$ and $C_0 C_i = 0$ for all $i \neq 0$. Let $A \coloneqq D - C_0$. Then $A \in K_X^\perp$ and $A^2 = 0$. But this implies $A = nK_X$ for some $n \in \mathbb{Z}$, since any isotropic vector in $K_X^\perp$ is a multiple of K_X. Indeed,

$$K_X, K_X - E_1, E_2 - E_3, \ldots, E_8 - E_9, H - 3E_9$$

is a basis of $\mathrm{Pic}(X)$ inducing an orthogonal decomposition

$$\mathrm{Pic}(X) = \langle K_X, K_X - E_1 \rangle \oplus \langle E_2 - E_3, \ldots, E_8 - E_9, H - 3E_9 \rangle.$$

The lattice $\langle E_2 - E_3, \ldots, E_8 - E_9, H - 3E_9 \rangle$ is even and negative definite, i.e., it is $E_8(-1)$, and the intersection form on $\langle K_X, K_X - E_1 \rangle$ is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

In particular, $K_X^\perp / \mathbb{Z}K_X$ is negative definite, so $A = nK_X$. Now $C_0 K_X = -1$ together with $C_0 A = 0$ implies $n = 0$ and hence $C_0 = D$. □

Remark 1.4.4 (On the position of the blown up points). The position of the 9 blown up points is important for only two facts: On the one hand we need to choose the position general enough so that there exists a unique cubic passing through the 9 points with multiplicity 1 and on the other hand in the proof of Lemma 1.4.3 we use the result of [Fer05] which depends on the position of the points. For the latter the assumptions are made more precise in [Fer05, Def. 2.1]. Alternatively one can replace [Fer05, Prop. 2.3] by [Nag60, Prop. 12] and assume that the points are in a position described in [Nag60, Prop. 9] to ensure that the surface carries no integral curve C with $C^2 \leq -2$.

Remark 1.4.5. We will see in Section 1.5 that the conclusion of Lemma 1.4.3 does not hold for blow-ups of 10 or more points.

1.4.2. Roots in the Picard lattice. Recall that any unimodular lattice Λ contains a root system with roots given by the elements $\alpha \in \Lambda$ such that $\alpha^2 = \pm 1$ or $\alpha^2 = \pm 2$. For such a root $\alpha \in \Lambda$ the reflection along $\alpha^\perp$ is given by

$$s_\alpha(x) \coloneqq x - 2\frac{(x \cdot \alpha)}{\alpha^2}\alpha.$$

Any such reflection is an orthogonal transformation of Λ. In particular, if Λ is even and negative-definite, then the roots of Λ are precisely the elements of square -2.

Let X be the blow-up of $\mathbb{P}^2$ in $n \geq 3$ points. Let H be the pullback of a hyperplane class and let $E_1, \dots, E_n$ be the exceptional divisors. Then $H, E_1, \dots, E_n$ is an orthogonal basis of the Picard lattice $\mathrm{Pic}(X)$ such that $H^2 = 1$ and $E_i^2 = -1$. The elements $\alpha_1 := E_1 - E_2, \dots, \alpha_{n-1} := E_{n-1} - E_n$ and $\alpha_0 := H - E_1 - E_2 - E_3$ are roots in $\mathrm{Pic}(X)$ and we denote by W_X the reflection group generated by s_{α_i}, $0 \leq i \leq n-1$. All roots α_i lie in $K_X^\perp$, thus

$$W_X \subseteq \mathrm{O}(\mathrm{Pic}(X))_{K_X} \subseteq \mathrm{O}(\mathrm{Pic}(X)),$$

where $\mathrm{O}(\mathrm{Pic}(X))_{K_X}$ is the stabilizer of the canonical class $K_X = -3H + \sum_i E_i$.

Lemma 1.4.6. *Let X be the blow-up of $\mathbb{P}^2$ in n points, where $3 \leq n \leq 9$. Then the reflection group $W_X = \langle s_{\alpha_0}, \dots, s_{\alpha_{n-1}} \rangle$ equals the stabilizer $\mathrm{O}(\mathrm{Pic}(X))_{K_X}$.*

Proof. First note that the equality $W_X = \mathrm{O}(\mathrm{Pic}(X))_{K_X}$ does not depend on the position of points and not on the base field. Thus, we can assume that the points p_i lie in very general position, $-K_X$ is class of an irreducible reduced curve in X, and that the base field is $\mathbb{C}$. Let $\sigma\colon X \to \mathbb{P}^2_{\mathbb{C}}$ be the blow-up morphism and denote by $E_i \subseteq X$ be the exceptional divisor over the point p_i Then Lemma 1.4.3 (or [EL16, Lem. 3.4] if $n \leq 8$) implies that any orthogonal transformation in $\varphi \in \mathrm{O}(\mathrm{Pic}(X))_{K_X}$ maps (-1)-curves to (-1)-curves. In particular, $E_1, \dots, E_n$ is mapped to an orthogonal set of (-1)-curves $E_1', \dots, E_n'$ and the class H of a hyperplane pulled back along σ is mapped to

$$H' := \varphi(H) = \frac{1}{3}\varphi\left(-K_X + \sum_{i=1}^n E_i\right) = \frac{1}{3}\left(-K_X + \sum_{i=1}^n E_i'\right).$$

By [Har85, Thm. 0.1], which is essentially a reformulation of results in [Nag60], we obtain $\varphi \in W_X$. Alternatively, $\varphi \in W_X$ follows from [DO88, Ch. VI, Thm. 2]. ☐

1.4.3. A weak del Pezzo surface admitting a numerically exceptional collection of maximal length which is not exceptional. We cannot expect that the conclusion of Lemma 1.4.3 holds true for rational surface of higher Picard rank, as we show in Section 1.5. But already if we blow up less than 9 points in special position, the conclusion of Lemma 1.4.3 does not hold. As a consequence, in general a maximal numerically exceptional collection does not need to be exceptional. In Proposition 1.4.7 we construct such an example by blowing up 8 points in a special position. Similar examples were already obtained for Hirzebruch surfaces Σ_d with even d in [EL16, Rmk. 2.18].

Proposition 1.4.7. *Let $\pi\colon X \to \mathbb{P}^2$ be the blow-up of 8 points $p_1, \dots, p_8$ such that p_1, p_2, p_3 lie on a line L and $p_4, \dots, p_8$ on a smooth irreducible conic curve C such that $p_1, p_2, p_3 \notin C$ and $p_4, \dots, p_8 \notin L$. Then X is a weak del Pezzo surface, i.e., $-K_X$ is nef and big, but admits a maximal numerically exceptional collection consisting of line bundles which is not exceptional. Moreover, X admits an effective divisor D satisfying $D^2 = -1$, $\chi(D) = 1$ and $H^1(X, \mathcal{O}_X(-D)) \neq 0$.*

Proof. Denote by E_i the exceptional divisor corresponding to the blow-up of p_i and let H be the pullback of the hyperplane class in $\mathbb{P}^2$. Then the anticanonical divisor satisfies $-K_X = 3H - \sum_{i=1}^8 E_i$ and thus is equal to the sum of the strict transform $\tilde{L}$ of L and the strict transform $\tilde{C}$ of C. Hence, the intersection of $-K_X$ with any other curve is non-negative and one checks that $-K_X\tilde{L} = 0$ is zero and $-K_X\tilde{C} = 1$. Therefore $-K_X$ is nef and hence $(-K_X)^2 = 1 > 0$ implies that $-K_X$ is big. Consider the divisor

$$D := 4H - 2E_1 - 2E_2 - 2E_3 - E_4 - \cdots - E_8 = 2\tilde{L} + \tilde{C},$$

which satisfies $D^2 = -1$ and $-K_X D = 1$. Then D is effective, but since $D \cdot \tilde{L} = -2$, any divisor in the linear system $|D|$ contains $\tilde{L}$ as an irreducible component. Thus, D cannot be the class of a (-1)-curve. Further we compute $h^1(-D) = 1$: Riemann–Roch yields $\chi(-D) = 0$ and since D is effective, $-D$ admits no global sections. This gives $h^1(-D) = h^2(-D) = h^0(K_X + D) = h^0(H - E_1 - E_2 - E_3) = 1$. Thus the conclusion of Lemma 1.4.3 does not hold for D.

Finally, we complete D into a toric system in order to obtain a maximal numerically exceptional collection consisting of line bundles. The set of orthogonal transformations of $\mathrm{Pic}(X)$ fixing the canonical class K_X coincides with the orthogonal group of $K_X^{\perp}$. A computation shows that $K_X^{\perp}$ identifies with the $E_8(-1)$-lattice and therefore the orthogonal group is the Weyl group of E_8. It is known that the Weyl group acts transitively on the set of roots; see, e.g., [Hum78, § 10.4 Lem. C]. Moreover, the map $F \mapsto F - K_X$ defines a bijection

$$\{\text{roots in}\,\mathrm{Pic}(X)\} \xrightarrow{\sim} \{F \in \mathrm{Pic}(X) \mid F^2 = K_X F = -1\},$$

see, e.g., [Dol12, § 8.1.1]. Hence, there exists an orthogonal transformation T of $\mathrm{Pic}(X)$ which fixes K_X and sends E_1 to D. Therefore, the image of the toric system associated to

$$\mathsf{D}^b(X) = \langle \mathcal{O}_X, \mathcal{O}_X(E_1), \ldots, \mathcal{O}_X(E_8), \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle$$

under T is a toric system which corresponds to a maximal numerically exceptional collection consisting of line bundles, which is not exceptional since $H^1(X, \mathcal{O}_X(D)) \neq 0$. □

1.4.4. Towards Theorem 1.1.3. The proof of Theorem 1.1.3 is separated in two steps. Recall that [EL16, Thm. 3.1] states that, on a del Pezzo surface, any toric system is obtained from a sequence of augmentations from an exceptional toric system on $\mathbb{P}^2$ or a Hirzebruch surface. In the first step, we generalize this result to the blow-up X of 9 very general points. In the second step, we prove the transitivity of the braid group action, as stated in Theorem 1.1.3, by realizing each orthogonal transformation of $\mathrm{Pic}(X)$ fixing K_X as a sequence of mutations.

The following Lemma 1.4.8 ensures that we can reduce X to a del Pezzo surface by contracting any (-1)-curve. As we were unable to find a suitable statement in the literature, we include a proof.

Lemma 1.4.8. *Let X be the blow-up of $\mathbb{P}^2$ in 9 very general points and let $E \subseteq X$ be a (-1)-curve. Then the surface Y obtained from blowing down E is a del Pezzo surface.*

Proof. Recall that the blow-up of at most 8 points in $\mathbb{P}^2$ is a del Pezzo surface if and only if not 3 of the points lie on a line, not 6 lie on a conic, and not 8 of them on a cubic having a node at one of them. Therefore the points are in special position if and only if the surface admits a $(-k)$-curve with $k \geq 2$, namely the strict transform of the cubic through 8 blown up points with a node at one of them, the conic through 6 blown up points, or the line through 3 blown up points. We further observe that the equivalence also holds true if the points are chosen infinitely near: If a point p is blown up on an exceptional divisor E, then the class of the strict transform of E is $E - E_p$, where E_p is the exceptional divisor corresponding to the blow-up of p. We compute $(E - E_p)^2 = -2$ in that case.

Let Y be the blow-down of the (-1)-curve and $\pi\colon X \to Y$ the blow-up map with center $p \in Y$ and exceptional divisor E. Then for any curve C in Y, the strict transform in X has divisor class $p^*C - mE$, where m is the multiplicity of C at p. Thus the self-intersection of the strict transform of C is $C^2 - m^2$. By [Fer05, Prop. 2.3], X has no integral curves of self-intersection ≥ 2, hence Y has no $(-k)$-curves with $k \geq 2$. Now Y is obtained from $\mathbb{P}^2$ by a sequence of blow-ups of (possibly infinitely near) 8 points. As Y has no $(-k)$-curves with $k \geq 2$, Y must be a del Pezzo surface. □

Theorem 1.4.9. *Let X be the blow-up of $\mathbb{P}^2$ in 9 very general points. Any toric system on X of length 12 is a standard augmentation, i.e., it is obtained by a sequence of augmentations from a full exceptional toric system on $\mathbb{P}^2$ or from a full exceptional toric system on a (non necessarily minimal) Hirzebruch surface.*

Proof. Let $A_1, \dots, A_{12}$ be a toric system on X. By Riemann–Roch, we have $\chi(A_i) = 2 + A_i^2$ for all $1 \leq i \leq 12$. By Lemma 1.4.3, Lemma 1.4.8, and [EL16, Thm. 3.1] we only need to show that there is a divisor A_i with $A_i^2 = -1$. In this situation the argument of Elagin–Lunts still applies: By [HP11, Prop. 2.7] there exists a smooth toric surface Y with torus invariant irreducible divisors $D_1, \dots, D_{12}$ such that $D_i^2 = A_i^2$ for any i. Since Y is not minimal, Y contains (-1)-curve which must be torus invariant as otherwise the self-intersection would be non-negative. We conclude that one of the D_i squares to -1, hence there exists A_i with $A_i^2 = -1$. □

Corollary 1.4.10. *On the blow-up of $\mathbb{P}^2$ in 9 very general points any numerically exceptional collection of maximal length consisting of line bundles is a full exceptional collection.*

Proof. By [EL16, Prop. 2.21] a standard augmentation corresponds to a full exceptional collection. □

In order to conclude the proof of Theorem 1.1.3 we are left to show that any two full exceptional collections resulting from two different sequences of augmentations are related by mutations and shifts. On a del Pezzo surface, an exceptional object is completely determined by its class in the Grothendieck group:

Lemma 1.4.11 (Exceptional objects on del Pezzo surfaces, [Gor88; KO94])**.** *Let X be a del Pezzo surface and let $E \in \mathsf{D}^b(X)$ be an exceptional object. Then E is isomorphic to some $F[k]$, where F is an exceptional sheaf on X and $k \in \mathbb{Z}$. Moreover F is either locally free or a torsion sheaf of the form $\mathcal{O}_C(d)$, where C is a (-1)-curve. In particular, two exceptional objects with the same image in $\mathsf{K}_0(X)$ only differ by an even number of shifts.*

Pointer to references. That every exceptional object is a sheaf up to shift can be found in [KO94, Prop. 2.10] and [KO94, Prop. 2.9] states that an exceptional sheaf is locally free or a torsion sheaf of the form $\mathcal{O}_C(d)$ where C is a (-1)-curve. In the latter case, such torsion sheaf is clearly uniquely determined by its Chern character and hence by its class in $\mathsf{K}_0(X)$. The case of locally free sheaves is treated in [Gor88, Cor. 2.5]. □

For later use in the proof of Theorem 1.4.18 we compute in the following Lemma 1.4.12 a relation by mutations and shifts of two concrete exceptional collections on the blow-up of $\mathbb{P}^2$ in 3 points. The statement of Lemma 1.4.12 can also be deduced from [KO94, Thm. 7.7]. We give an independent proof by computing an explicit sequence of mutations relating both collections.

Lemma 1.4.12. *Let X be the blow-up of 3 points in $\mathbb{P}^2$ which do not lie on a line. Then the full exceptional collections*

$$\mathsf{D}^b(X) = \langle \mathcal{O}_{E_1}(-1), \mathcal{O}_{E_2}(-1), \mathcal{O}_{E_3}(-1), \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle$$

and

$$\begin{aligned} \mathsf{D}^b(X) = \langle &\mathcal{O}_{H-E_2-E_3}(-1), \mathcal{O}_{H-E_1-E_3}(-1), \mathcal{O}_{H-E_1-E_2}(-1), \\ &\mathcal{O}_X, \mathcal{O}_X(2H - E_1 - E_2 - E_3), \mathcal{O}_X(4H - 2E_1 - 2E_2 - 2E_3) \rangle \end{aligned}$$

are related by mutations and shifts.

Proof. Since X is a del Pezzo surface it is enough to verify the claim in $\mathsf{K}_0(X)$ by using Lemma 1.4.11. In $\mathsf{K}_0(X)$ this becomes a lattice-theoretic computation: Let

$$a_i := [\mathcal{O}_{E_i}(-1)] \text{ and } b_1 := [\mathcal{O}_X], b_2 := [\mathcal{O}_X(H)], b_3 := [\mathcal{O}_X(2H)].$$

Then the Gram matrix corresponding to the basis $(a_1, a_2, a_3, b_1, b_2, b_3)$ is

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.4.13}$$

Similarly to (1.3.8) we have the following sequence of mutations

$$\begin{aligned}(a_1, a_2, a_3, b_1, b_2, b_3) &\xrightarrow{\mathsf{L}_{5,6}}\xrightarrow{\mathsf{R}_{3,4}}\xrightarrow{\mathsf{R}_{2,3}}\xrightarrow{\mathsf{L}_{4,5}}\xrightarrow{\mathsf{L}_{3,4}}\xrightarrow{\mathsf{R}_{2,3}}\xrightarrow{\mathsf{R}_{1,2}}\xrightarrow{\mathsf{L}_{2,3}}\xrightarrow{\mathsf{R}_{3,4}}\xrightarrow{\mathsf{R}_{4,5}}\xrightarrow{\mathsf{L}_{2,3}}\xrightarrow{\mathsf{L}_{3,4}}\xrightarrow{\mathsf{R}_{5,6}} \\ &(a_2 + a_3 + 2b_1 - 3b_2 + b_3, -a_1 - a_3 - 2b_1 + 3b_2 - b_3, -a_1 - a_2 - 2b_1 + 3b_2 - b_3, \\ &\qquad a_1 + a_2 + a_3 + 3b_1 - 3b_2 + b_3, b_2, a_1 + a_2 + a_3 + 2b_1 - 3b_2).\end{aligned}$$

After changing the sign of the first and last basis elements we obtain the exceptional basis

$$\begin{aligned}&(-a_2 - a_3 - 2b_1 + 3b_2 - b_3, -a_1 - a_3 - 2b_1 + 3b_2 - b_3, -a_1 - a_2 - 2b_1 + 3b_2 - b_3, \\ &\qquad a_1 + a_2 + a_3 + 3b_1 - 3b_2 + b_3, b_2, -a_1 - a_2 - a_3 - 2b_1 + 3b_2),\end{aligned} \tag{1.4.14}$$

which has Gram matrix (1.4.13). Recall that the Chern character on a surface is given by

$$\mathrm{ch} = \left(\mathrm{rk}, \mathrm{c}_1, \frac{1}{2}(\mathrm{c}_1^2 - 2\,\mathrm{c}_2)\right).$$

We compute

$$\begin{aligned}\mathrm{ch}(a_i) &= \mathrm{ch}(\mathcal{O}_{E_i}(E_i)) = \left(0, E_i, -\frac{1}{2}\right), \\ \mathrm{ch}(b_1) &= \mathrm{ch}(\mathcal{O}_X) = (1, 0, 0), \\ \mathrm{ch}(b_2) &= \mathrm{ch}(\mathcal{O}_X(H)) = \left(1, H, \frac{1}{2}\right), \\ \mathrm{ch}(b_3) &= \mathrm{ch}(\mathcal{O}_X(2H)) = (1, 2H, 2).\end{aligned}$$

Thus (1.4.14) corresponds to the full exceptional collection

$$\langle \mathcal{O}_{H-E_2-E_3}, \mathcal{O}_{H-E_1-E_3}, \mathcal{O}_{H-E_1-E_2}, \mathcal{O}_X(-H+E_1+E_2+E_3), \mathcal{O}_X(H), \mathcal{O}_X(3H-E_1-E_2-E_3)\rangle. \tag{1.4.15}$$

Here we denote, in abuse, the strict transform of the line through the points p_i and p_j by $H - E_i - E_j$. Note that K_X can be rewritten as

$$\begin{aligned}K_X &= -3H + E_1 + E_2 + E_3 \\ &= -3(2H - E_1 - E_2 - E_3) + (H - E_2 - E_3) + (H - E_1 - E_3) + (H - E_1 - E_2),\end{aligned}$$

where $2H - E_1 - E_2 - E_3$ can be identified with the pullback of a hyperplane class on $\mathbb{P}^2$ considered as the blow-down of $(H - E_2 - E_3)$, $(H - E_1 - E_3)$ and $(H - E_1 - E_2)$. Hence

$$\mathcal{O}_{H-E_i-E_j}(K_X) = \mathcal{O}_{H-E_i-E_j}(H - E_i - E_j) = \mathcal{O}_{H-E_i-E_j}(-1),$$

where we have used the projection formula in the first equality. Recall that any twist with an integer multiple of the canonical line bundle can be realized as a sequence of mutations. Twisting (1.4.15) by K_X yields

$$\langle \mathcal{O}_{H-E_2-E_3}(-1), \mathcal{O}_{H-E_1-E_3}(-1), \mathcal{O}_{H-E_1-E_2}(-1),\\ \mathcal{O}_X(-4H+2E_1+2E_2+2E_3), \mathcal{O}_X(-2H+E_1+E_2+E_3), \mathcal{O}_X\rangle.$$

Finally, by applying the sequence $\mathsf{R}_{5,6}\circ\mathsf{R}_{4,5}\circ\mathsf{R}_{5,6}\circ\mathsf{R}_{4,5}$ of mutations, we obtain the desired full exceptional collection

$$\mathsf{D}^b(X) = \langle \mathcal{O}_{H-E_2-E_3}(-1), \mathcal{O}_{H-E_1-E_3}(-1), \mathcal{O}_{H-E_1-E_2}(-1),\\ \mathcal{O}_X, \mathcal{O}_X(2H-E_1-E_2-E_3), \mathcal{O}_X(4H-2E_1-2E_2-2E_3)\rangle. \qquad \square$$

Remark 1.4.16 (Alternate proof of Lemma 1.4.12). Alexander Kuznetsov pointed out to us that Lemma 1.4.12 can also be proven by the following sequence of mutations:

$$\langle \mathcal{O}_{E_1}(-1), \mathcal{O}_{E_2}(-1), \mathcal{O}_{E_3}(-1), \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle$$
$$\xrightarrow{\mathsf{R}_{3,4}\circ\mathsf{R}_{2,3}\circ\mathsf{R}_{1,2}\circ\mathsf{R}_{4,5}\circ\mathsf{R}_{3,4}\circ\mathsf{R}_{2,3}\circ\mathsf{R}_{5,6}\circ\mathsf{R}_{4,5}\circ\mathsf{R}_{3,4}}$$
$$\langle \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H), \mathcal{O}_{E_1}, \mathcal{O}_{E_2}, \mathcal{O}_{E_3}\rangle$$
$$\xrightarrow{\mathsf{R}_{5,6}\circ\mathsf{R}_{4,5}\circ\mathsf{R}_{3,4}}$$
$$\langle \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_{E_1}, \mathcal{O}_{E_2}, \mathcal{O}_{E_3}, \mathcal{O}_X(2H-E_1-E_2-E_3)\rangle$$
$$\xrightarrow{\mathsf{L}_{4,5}\circ\mathsf{L}_{3,4}\circ\mathsf{L}_{2,3}}$$
$$\langle \mathcal{O}_X, \mathcal{O}_X(H-E_1), \mathcal{O}_X(H-E_2), \mathcal{O}_X(H-E_3), \mathcal{O}_X(H), \mathcal{O}_X(2H-E_1-E_2-E_3)\rangle$$

The same sequence of mutations applied to

$$\langle \mathcal{O}_{H-E_2-E_3}(-1), \mathcal{O}_{H-E_1-E_3}(-1), \mathcal{O}_{H-E_1-E_2}(-1),\\ \mathcal{O}_X, \mathcal{O}_X(2H-E_1-E_2-E_3), \mathcal{O}_X(4H-2E_1-2E_2-2E_3)\rangle$$

yields

$$\langle \mathcal{O}_X, \mathcal{O}_X(H-E_1), \mathcal{O}_X(H-E_2), \mathcal{O}_X(H-E_3), \mathcal{O}_X(2H-E_1-E_2-E_3), \mathcal{O}_X(H)\rangle.$$

Since $\mathcal{O}_X(2H-E_1-E_2-E_3)$ and $\mathcal{O}_X(H)$ are orthogonal, this proves the lemma.

Remark 1.4.17 (Geometric interpretation of Lemma 1.4.12). The surface X in Lemma 1.4.12 admits two different blow-up realizations. First one blows up 3 points p_1, p_2, p_3 in $\mathbb{P}^2$ and then one contract the (-1)-curves $H - E_i - E_j$ which are the strict transforms of the lines through the points p_i, p_j. The full exceptional collections compared in Lemma 1.4.12 are the collections resulting from Orlov's blow-up formula applied to these different realizations of X. Moreover, this construction defines a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, which is known as *standard quadratic Cremona transformation.*

Theorem 1.4.18. *On the blow-up X of $\mathbb{P}^2$ in 9 very general points, any two full exceptional collections consisting of line bundles are related by mutations and shifts.*

Proof. For the sake of simplicity we call two full exceptional collections *equivalent* if they can be transformed into each other by a sequence of mutations and shifts. Let

$$\mathsf{D}^b(X) = \langle \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_n)\rangle, \tag{1.4.19}$$

where $n = 12$, be a full exceptional collection consisting of line bundles. Then $\langle \mathcal{O}_X(D_2), \ldots, \mathcal{O}_X(D_n), \mathcal{O}_X(D_1 - K_X)\rangle$ is an equivalent collection; see, e.g., [GK04, Cor. 2.8.3]. In particular, any twist with an integer multiple of the canonical class can be realized as a sequence of mutations and shifts. By Theorem 1.4.9 the toric system associated to

(1.4.19) contains a (-1)-curve. After passing to an equivalent collection, we can assume that $E := D_2 - D_1$ is a (-1)-curve. The left mutation of the pair $\langle \mathcal{O}_X(D_1), \mathcal{O}_X(D_2)\rangle$ is defined by the exact triangle

$$\mathcal{O}_X(D_1) \otimes \mathrm{RHom}(\mathcal{O}_X(D_1), \mathcal{O}_X(D_2)) \xrightarrow{\mathrm{ev}} \mathcal{O}_X(D_2) \to \mathsf{L}_{\mathcal{O}_X(D_1)}(\mathcal{O}_X(D_2)).$$

On the other hand

$$\mathrm{RHom}(\mathcal{O}_X(D_1), \mathcal{O}_X(D_2)) = H^\bullet(X, \mathcal{O}_X(E)) = \mathbb{C}[0].$$

Therefore the ideal sheaf sequence

$$0 \to \mathcal{O}_X(-E) \xrightarrow{\phi} \mathcal{O}_X \to \mathcal{O}_E \to 0$$

yields an exact triangle

$$\mathcal{O}_X(D_1) = \mathcal{O}_X(D_1) \otimes \mathrm{RHom}(\mathcal{O}_X(D_1), \mathcal{O}_X(D_2)) \xrightarrow{\mathrm{ev}} \mathcal{O}_X(D_2) \to \mathcal{O}_E(D_2),$$

where one verifies that $\phi \otimes \mathcal{O}_X(D_2)$ coincides with the evaluation map. As E is isomorphic to a projective line, we conclude that (1.4.19) is equivalent to

$$\mathsf{D}^b(X) = \langle \mathcal{O}_E(d), \mathcal{O}_X(D_1), \mathcal{O}_X(D_3), \dots, \mathcal{O}_X(D_n)\rangle \tag{1.4.20}$$

for some $d \in \mathbb{Z}$.

Let $p\colon X \to X'$ be the blow-down of E; then $K_X = p^*K_{X'} + E$. Using the projection formula to compute $\mathcal{O}_E(d) \otimes \mathcal{O}_X(K_X) = \mathcal{O}_E(d) \otimes \mathcal{O}_X(E)$, we can assume that $d = -1$ by twisting (1.4.20) with $(d+1)K_X$. This means that (1.4.19) is equivalent to a full exceptional collection obtained by the blow-up formula from a del Pezzo surface X'. By using Theorem 1.3.1, we can mutate the full exceptional collection

$$\langle \mathcal{O}_X(D_1), \mathcal{O}_X(D_3), \dots, \mathcal{O}_X(D_n)\rangle = \mathsf{D}^b(X')$$

to a collection consisting of 8 rank 0 and 3 rank 1 objects. By Lemma 1.4.11, the rank 0 objects are (shifts of) torsion sheaf on (-1)-curves. Thus, we can assume that the full exceptional collection on X' comes from iterated blow-ups of a copy of $\mathbb{P}^2$. Hence, (1.4.20) is equivalent to a collection

$$\langle \mathcal{O}_{E_1'}(-1), \mathcal{O}_{E_2'}(-d_2), \dots, \mathcal{O}_{E_9'}(-d_9), \mathcal{O}_X(nH'), \mathcal{O}_X((n+1)H'), \mathcal{O}_X((n+2)H')\rangle,$$

where $E_1', \dots, E_9'$ are pairwise disjoint (-1)-curves with $E_i'H' = 0$ and $H'^2 = 1$. Twisting the partial sequence

$$\langle \mathcal{O}_{E_2'}(-d_2), \dots, \mathcal{O}_{E_9'}(-d_9), \mathcal{O}_X(nH'), \mathcal{O}_X((n+1)H'), \mathcal{O}_X((n+2)H')\rangle$$

with $K_{X'}$ can be realized as a sequence of mutations, because $(- \otimes \mathcal{O}_X(K_{X'})[2])$ is the Serre functor of

$$\langle \mathcal{O}_{E_2'}(-d_2), \dots, \mathcal{O}_{E_9'}(-d_9), \mathcal{O}_X(nH'), \mathcal{O}_X((n+1)H'), \mathcal{O}_X((n+2)H')\rangle \cong \mathsf{D}^b(X').$$

Thus we can assume $d_2 = 1$ and repeating this procedure, we can assume that $d_i = 1$ for all i. We have an equivalence $\langle \mathcal{O}_X(nH'), \mathcal{O}_X((n+1)H'), \mathcal{O}_X((n+2)H')\rangle \cong \mathsf{D}^b(\mathbb{P}^2)$, where H' is identified with a hyperplane class. On $\mathbb{P}^2$ we compute that $\langle \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(H), \mathcal{O}_{\mathbb{P}^2}(2H)\rangle$ is equivalent to

$$\langle \mathcal{O}_{\mathbb{P}^2}(H), \mathcal{O}_{\mathbb{P}^2}(2H), \mathcal{O}_{\mathbb{P}^2}(-K_{\mathbb{P}^2}) = \mathcal{O}_{\mathbb{P}^2}(3H)\rangle,$$

thus in our situation we can assume that $n = 0$. Therefore $E_1', \dots, E_9', H'$ can be obtained from $E_1, \dots, E_9, H$ by applying an orthogonal transformation of $\mathrm{Pic}(X)$ fixing the canonical class $-3H + \sum_i E_i = -3H' + \sum_i E_i'$.

It remains to show that the two sequences

$$\mathsf{D}^b(X) = \langle \mathcal{O}_X, \mathcal{O}_X(E_1), \dots, \mathcal{O}_X(E_9), \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle \text{ and}$$
$$\mathsf{D}^b(X) = \langle \mathcal{O}_X, \mathcal{O}_X(E'_1), \dots, \mathcal{O}_X(E'_9), \mathcal{O}_X(H'), \mathcal{O}_X(2H')\rangle$$

are equivalent. By Lemma 1.4.6, the group $\mathrm{O}(\mathrm{Pic}(X))_{K_X}$ coincides with the Weyl group generated by the reflections induced by the simple roots $E_1 - E_2, \dots, E_8 - E_9$, and $H - E_1 - E_2 - E_3$. The reflection along the hyperplane orthogonal to a (-2)-class v is given by

$$s_v(x) = x + (x \cdot v)v.$$

Thus if $v = E_i - E_{i+1}$, then s_v fixes $E_1, \dots E_{i-1}, E_{i+2}, \dots, H$ and permutes E_i and E_{i+1}. This can be identified with a mutation of the exceptional pair $\langle \mathcal{O}_X(E_i), \mathcal{O}_X(E_{i+1})\rangle$. Assume $v = H - E_1 - E_2 - E_3$; then s_v fixes $E_4, \dots, E_9$. Computing the corresponding mutation (on the blow-up of 3 points for simplicity) one observes that the full exceptional collection

$$\mathsf{D}^b(X) = \langle \mathcal{O}_X, \mathcal{O}_X(E_1), \mathcal{O}_X(E_2), \mathcal{O}_X(E_3), \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle$$

is changed to

$$\begin{aligned}\mathsf{D}^b(X) = \langle \mathcal{O}_X, \mathcal{O}_X(H - E_2 - E_3), \mathcal{O}_X(H - E_1 - E_3), \mathcal{O}_X(H - E_1 - E_2),\\ \mathcal{O}_X(2H - E_1 - E_2 - E_3), \mathcal{O}_X(4H - 2E_1 - 2E_2 - 2E_3)\rangle.\end{aligned}$$

This is the full exceptional collection obtained by the blow-up formula after blowing down the strict transforms of the lines through 2 of the points. By Lemma 1.4.12 this simple reflection can also be realized as a sequence of mutations and shifts.

In general an element of the Weyl group is a composition of simple reflections $s_{v_1} \circ \cdots \circ s_{v_m}$. Recall that for reflections $s_v \circ s_w \circ s_v = s_{s_v(w)}$ holds. This gives

$$s_{s_v(w)} \circ s_v = s_v \circ s_w.$$

Applying this to our composition of simple reflections we can rewrite

$$s_{v_1} \circ \cdots \circ s_{v_m} = s_{s_{v_1}(v_2)} \circ \cdots \circ s_{s_{v_1}(v_m)} \circ s_{v_1}.$$

We conclude now by induction: After realizing s_{v_1} by mutations and shifts, $s_{s_{v_1}(v_2)} \circ \cdots \circ s_{s_{v_1}(v_m)}$ is a sequence of $m-1$ simple reflections with respect to the new basis of simple roots obtained after applying s_{v_1}. Hence it can be realized as a sequence of mutations and shifts. □

We immediately obtain:

Corollary 1.4.21. *On the blow-up X of $\mathbb{P}^2$ in 9 very general points, twisting a full exceptional collections consisting of line bundles by a line bundle can be realized as a sequence of mutations and shifts.*

As a further corollary we obtain a new proof of a result of Kuleshov–Orlov.

Corollary 1.4.22 (cf. [KO94, Thm. 7.7])**.** *Let X be a del Pezzo surface, then any two full exceptional collections on X are related by mutations and shifts.*

Proof. Recall that X is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blow-up of less than 8 points in $\mathbb{P}^2$ in general position. Given the latter case, suppose $E_\bullet$ and $F_\bullet$ are two full exceptional collections on X. By Theorem 1.3.1 we can assume that $E_\bullet$ and $F_\bullet$ consist of rank 1 objects. Now by Lemma 1.4.11 exceptional rank 1 objects on X are line bundles and we argue as in the proof of Theorem 1.4.18.

Assume $X = \mathbb{P}^1 \times \mathbb{P}^1$, then $\mathrm{Pic}(X) = \mathbb{Z}H_1 \oplus \mathbb{Z}H_2$ with $H_1H_2 = 1$ and $H_1^2 = H_2^2 = 0$ and $K_X = -2H_1 - 2H_2$. One computes that any $D \in \mathrm{Pic}(X)$ such that $D^2 = 0$ is either a multiple of H_1 or a multiple of H_2. In particular, $D = H_1$ or $D = H_2$ are the only classes such that $D^2 = 0$ and $K_X D = -2$. Thus, the orthogonal transformations of $\mathrm{Pic}(X)$ fixing K_X are exactly the permutations of H_1 and H_2. Let $E_\bullet$ be a full exceptional collection on X. As before we can assume that $E_\bullet$ is a sequence consisting of line bundles. By Theorem 1.3.1 and Lemma 1.2.11 $E_\bullet$ has the form

$$\langle \mathcal{O}_X(a,b), \mathcal{O}_X(a+1,b), \mathcal{O}_X(a,b+1), \mathcal{O}_X(a+1,b+1)\rangle \text{ or}$$
$$\langle \mathcal{O}_X(a,b), \mathcal{O}_X(a,b+1), \mathcal{O}_X(a+1,b), \mathcal{O}_X(a+1,b+1)\rangle.$$

Both are equivalent as the mutation $\mathsf{L}_{2,3}$ permutes the middle factors. One computes that the right mutation $\mathsf{R}_{\mathcal{O}_X(a+1,b+1)}(\mathcal{O}_X(a,b+1))$ is equal to $\mathcal{O}_X(a+2,b+1)$ up to possible shifts and similarly $\mathsf{R}_{\mathcal{O}_X(a+1,b+1)}(\mathcal{O}_X(a+1,b))$ identifies with $\mathcal{O}_X(a+1,b+2)$. We deduce that $E_\bullet$ is equivalent to

$$\langle \mathcal{O}_X(a,(b+1)), \mathcal{O}_X(a+1,(b+1)), \mathcal{O}_X(a,(b+1)+1), \mathcal{O}_X(a+1,(b+1)+1)\rangle,$$

hence we realized the twist by $\mathcal{O}_X(0,1)$ as a sequence of mutations. Analogously one obtains that the twist by $\mathcal{O}_X(1,0)$ is a sequence of mutations and therefore $E_\bullet$ is equivalent to

$$\langle \mathcal{O}_X(0,0), \mathcal{O}_X(1,0), \mathcal{O}_X(0,1), \mathcal{O}_X(1,1)\rangle. \qquad \square$$

Corollary 1.4.23. *Let X be a smooth projective surface over a field k with $\chi(\mathcal{O}_X) = 1$, $K_X^2 + \mathrm{rk}(\mathsf{K}_0^{\mathrm{num}}(X)) = 12$, and $\mathrm{rk}\,\mathsf{K}_0^{\mathrm{num}}(X) \leq 12$. Then any two exceptional bases $e_\bullet$ and $f_\bullet$ of $\mathsf{K}_0^{\mathrm{num}}(X)$ are related by a sequence of mutations and sign changes.*

Proof. By Vial's classification, see Theorem 1.2.13 and Remark 1.2.14, we can assume that X is a del Pezzo surface or the blow-up of $\mathbb{P}^2$ in 9 points. In these cases $\mathsf{K}_0^{\mathrm{num}}(X)$ is independent from the base field and the position of points, thus we can assume that the base field is $\mathbb{C}$ and the blown up points are in very general position. Moreover, by Perling's algorithm, see [Per18, Thm. 10.9], we can assume that $e_\bullet$ and $f_\bullet$ only consist of rank 1 objects. Recall that for a numerically exceptional object $E \in \mathsf{D}^b(X)$ the Riemann–Roch formula implies $\mathrm{c}_2(E) = 0$, thus we may assume that $e_\bullet$ and $f_\bullet$ arise from two numerically exceptional collections of maximal length consisting of line bundles. By Corollary 1.4.10 if $\mathrm{rk}\,\mathsf{K}_0^{\mathrm{num}}(X) = 12$ or [EL16, Thm. 3.1] if $\mathrm{rk}\,\mathsf{K}_0^{\mathrm{num}}(X) \leq 11$, $e_\bullet$ and $f_\bullet$ arise from full exceptional collections consisting of line bundles. Hence, the corollary follows from Theorem 1.4.18 and Corollary 1.4.22. $\square$

1.5. Blow-up of 10 Points

Although the situation of 9 blown up points is similar to the case of del Pezzo surfaces, the situation changes if we blow up 10 points. In fact, the conclusions of Lemma 1.4.3 and Lemma 1.4.6 do not hold for the blow-up of 10 points.

Lemma 1.5.1. *Let X be the blow-up of $\mathbb{P}^2$ in 10 points. Then the stabilizer of the canonical class is*

$$\mathrm{O}(\mathrm{Pic}(X))_{K_X} = W_X \times \langle \iota \rangle,$$

where W_X is the reflection group generated by the simple reflections corresponding to the roots $H - E_1 - E_2 - E_3, E_1 - E_2, \dots, E_9 - E_{10}$ and ι is the involution of $\mathrm{Pic}(X)$ fixing K_X and given by multiplication of -1 on $K_X^\perp$.

Proof. Denote the roots by $\alpha_0 := H - E_1 - E_2 - E_3, \alpha_1 := E_1 - E_2, \dots, \alpha_9 := E_9 - E_{10}$. Since $K_X^2 = -1$, $\mathrm{Pic}(X)$ splits as an orthogonal direct sum

$$\mathrm{Pic}(X) = K_X^\perp \oplus \mathbb{Z}K_X.$$

One can compute that a basis of $K_X^\perp$ is given by the roots $\alpha_0, \ldots, \alpha_9$. Since $\operatorname{Pic}(X)$ has signature $(1,10)$, $K_X^2 = -1$, and $\alpha_i^2 = -2$ for all $0 \le i \le 9$, $K_X^\perp$ is an even unimodular lattice of signature $(1,9)$. But it is known, that there is only one even unimodular lattice of signature $(9,1)$, which we denote by $\mathrm{II}_{9,1}$. Its orthogonal group was computed by Vinberg [Vin75]. Vinberg's result was rewritten by Conway–Sloane which use the description of $\mathrm{II}_{9,1}$ as the set

$$\left\{x = (x_0, \ldots, x_9) \in \mathbb{Z}^{10} \cup (\mathbb{Z}+1/2)^{10} \mid x_0 + \cdots + x_8 - x_9 \in 2\mathbb{Z}\right\} \subseteq \mathbb{Q}^{10}$$

with bilinear form $(x,y) := \sum_{i=0}^{8} x_i y_i - x_9 y_9$. Now [CS99, §27 Thm. 1] states that $\mathrm{O}(\mathrm{II}_{9,1}) = W_{\mathrm{II}_{9,1}} \times \{\pm \mathrm{id}_{\mathrm{II}_{9,1}}\}$, where $W_{\mathrm{II}_{9,1}}$ is the Weyl group of the root system in $\mathrm{II}_{9,1}$ with simple roots

$$\beta_i = (\underbrace{0, \ldots, 0}_{i}, 1, -1, \underbrace{0, \ldots, 0}_{8-i}) \text{ for } 0 \le i \le 7,$$
$$\beta_8 = (1/2, \ldots, 1/2), \text{ and } \beta_9 = (-1, -1, \underbrace{0, \ldots, 0}_{8}).$$

We observe that sending $\alpha_0 \mapsto \beta_0$, $\alpha_i \mapsto \beta_{i-2}$ for $3 \le i \le 9$, and $\alpha_i \mapsto \beta_{i+7}$ for $i = 1, 2$ yields a suitable isomorphism of lattices $K_X^\perp \xrightarrow{\sim} \mathrm{II}_{9,1}(-1)$ such that the α_i are send to the simple roots β_i. Clearly $\mathrm{O}(\operatorname{Pic}(X))_{K_X} = \mathrm{O}(K_X^\perp)$, thus the lemma follows. □

A further computation shows that

$$D_i := \iota(E_i) = -6H + 2\sum_{j=1}^{10} E_j - E_i \quad \text{and} \quad F := \iota(H) = -19H + 6\sum_{i=1}^{10} E_i.$$

Thus

$$\mathcal{O}_X, \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F)$$

is a numerically exceptional collection of maximal length on X. We show in Chapter 2 that the collection is exceptional but not full. The divisors D_i are not effective but satisfy $D_i^2 = -1$ and $\chi(D) = 1$. This shows that the conclusion of Lemma 1.4.3 does not hold for blow-ups of 10 or more points.

Proposition 1.5.2. *Let X be the blow-up of $\mathbb{P}^2$ in 10 general points. Then $\mathbb{Z}^{13} \rtimes \mathfrak{B}_{13}$, where $\mathfrak{B}_{13}$ is the braid group acting by mutations and $\mathbb{Z}^{13}$ acts by shifts, does not act transitively on the set of exceptional collections of length 13.*

Proof. Mutations and shifts do not change the generated subcategory of an exceptional collection. Thus the existence of a full and of a non-full exceptional collection of the same length shows that the action cannot be transitive. □

Proposition 1.5.3. *Let X be a smooth projective surface over a field k with $\chi(\mathcal{O}_X) = 1$ and $\operatorname{rk} \mathsf{K}_0^{\mathrm{num}}(X) = 13$ such that $\mathsf{K}_0^{\mathrm{num}}(X)$ admits an exceptional basis. Then the action of $\{\pm 1\}^{13} \rtimes \mathfrak{B}_{13}$ has at most 2 orbits.*

Proof. Without loss of generality, we assume that X is the blow-up of $\mathbb{P}^2$ in 10 general points. Applying Theorem 1.3.1, we know that each orbit contains an exceptional basis of the form

$$([\mathcal{O}_X(D_1)], \ldots, [\mathcal{O}_X(D_{13})]),$$

such that $D_2 - D_1 = \varphi(A_1), D_3 - D_2 = \varphi(A_2), \ldots, D_{13} - D_{12} = \varphi(A_{12})$, where $(A_1, \ldots, A_{13})$ is the toric system associated to the collection

$$\mathsf{D}^b(X) = \langle \mathcal{O}_X, \mathcal{O}_X(E_1), \ldots, \mathcal{O}_X(E_{10}), \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle$$

and $\varphi \in \mathrm{O}(\mathrm{Pic}(X))_{K_X}$. By Lemma 1.5.1, either $\varphi \in W_X$ or φ can be written as $\iota \circ w$ for some $w \in W_X$. Thus, it is enough to show that for each $\varphi \in W_X$ the collections

$$([\mathcal{O}_X(D_1)], \ldots, [\mathcal{O}_X(D_{13})]) \quad \text{and} \quad ([\mathcal{O}_X], [\mathcal{O}_X(E_1)], \ldots, [\mathcal{O}_X(E_{10})], [\mathcal{O}_X(H)], [\mathcal{O}_X(2H)])$$

lie in the same orbit. As $\varphi \in W_X$ sends (-1)-curves to (-1)-curves, $D_2 - D_1, \ldots, D_{11} - D_1$ is a set of disjoint (-1)-curves. Thus, we can argue as in Theorem 1.4.18 to reduce to showing that

$$\begin{aligned} &([\mathcal{O}_X], [\mathcal{O}_X(E_1)], \ldots, [\mathcal{O}_X(E_{10})], [\mathcal{O}_X(H)], [\mathcal{O}_X(2H)]) \\ \text{and } &([\mathcal{O}_X], [\mathcal{O}_X(\varphi(E_1))], \ldots, [\mathcal{O}_X(\varphi(E_{10}))], [\mathcal{O}_X(\varphi(H))], [\mathcal{O}_X(\varphi(2H))]) \end{aligned}$$

lie in the same orbit. But as φ can be factored in a sequence of simple reflections, this follows with the same argument as in Theorem 1.4.18. □

Remark 1.5.4. A characterization similar to Lemma 1.4.11 of exceptional objects in $\mathsf{D}^b(X)$, where X is the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 10 general points, would be of particular interest. More precisely, if one could verify condition (b) from Section 1.1 for exceptional collections of maximal length on X, one could conclude that there are 2 orbits of the $\{\pm 1\}^{13} \rtimes \mathfrak{B}_{13}$-action on exceptional bases of $\mathsf{K}_0^{\mathrm{num}}(X)$. One orbit would consist of the images of full exceptional collections and the the other orbit of the images of exceptional collections of length 13 which are not full.

CHAPTER 2

A Phantom on a Rational Surface

Based on [Kra24a]

SUMMARY. We construct a non-full exceptional collection of maximal length consisting of line bundles on the blow-up of the projective plane in 10 general points. As a consequence, the orthogonal complement of this collection is a universal phantom category. This provides a counterexample to a conjecture of Kuznetsov and to a conjecture of Orlov.

2.1. Introduction

Let X be a smooth projective variety over the field of complex numbers and denote by $\mathsf{D}^b(X)$ the bounded derived category of coherent sheaves on X. A nontrivial admissible subcategory $\mathcal{A} \subseteq \mathsf{D}^b(X)$ is called a *phantom* if the Grothendieck group $\mathsf{K}_0(\mathcal{A})$ vanishes. The first examples of phantom categories were constructed by Gorchinskiy–Orlov [GO13] and Böhning–Graf von Bothmer–Katzarkov–Sosna [BGKP15]. It follows from a result of Efimov that a so-called *universal phantom* can be embedded into a proper dg-category admitting a full exceptional collection [Efi23]; see Section 2.2 for the definition of a universal phantom. We provide a simple example of a variety which admits a full exceptional collection and a universal phantom subcategory.

Theorem 2.1.1. *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 10 general closed points $p_1, \ldots, p_{10} \in \mathbb{P}^2_{\mathbb{C}}$. Denote by H the divisor class obtained by pulling back the class of a hyperplane in $\mathbb{P}^2_{\mathbb{C}}$ and denote by E_i the class of the exceptional divisor over the point p_i, $1 \leq i \leq 10$. Then*

$$\langle \mathcal{O}_X, \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F) \rangle \subseteq \mathsf{D}^b(X), \tag{2.1.2}$$

$$\text{where} \quad D_i := -6H + 2\sum_{j=1}^{10} E_j - E_i \quad \text{and} \quad F := -19H + 6\sum_{i=1}^{10} E_i,$$

is an exceptional collection of maximal length which is not full.

It was previously shown in [Pir23, Thm. 6.35] that a del Pezzo surface Y does not admit a phantom in $\mathsf{D}^b(Y)$. Moreover, we showed in Chapter 1 that on the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 9 very general points every exceptional collection of maximal length consisting of line bundles is full. We discovered the exceptional collection (2.1.2) while trying to increase the number of blown up points in Theorem 1.1.3.

Since any blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of points admits a full exceptional collection, Theorem 2.1.1 disproves the following conjecture of Kuznetsov:

Conjecture 2.1.3 ([Kuz14, Conj. 1.10])**.** *Let $\mathcal{T} = \langle E_1, \ldots, E_n \rangle$ be a triangulated category generated by an exceptional collection. Then any exceptional collection of length n in $\mathcal{T}$ is full.*

As a consequence of Theorem 2.1.1, the right- or left-orthogonal complement of the collection (2.1.2) is a phantom category. In general, if $\mathcal{A}$ is an admissible subcategory of $\mathsf{D}^b(X)$ and $\mathsf{D}^b(X)$ admits a full exceptional collection, then by [Orl20, Cor. 3.4] $\mathcal{A}$ has a dg-enhancement quasi-equivalent to $\mathcal{P}erf{-}\mathcal{R}$, where $\mathcal{R}$ is a smooth finite-dimensional dg-algebra. Hence, Theorem 2.1.1 disproves the following conjecture of Orlov:

Conjecture 2.1.4 ([Orl20, Conj. 3.7]). *There are no phantoms of the form $\mathcal{P}erf{-}\mathcal{R}$, where $\mathcal{R}$ is a smooth finite-dimensional dg-algebra and $\mathcal{P}erf{-}\mathcal{R}$ is the dg-category of perfect dg-modules over $\mathcal{R}$.*

Recently, Chang–Haiden–Schroll gave an example of a triangulated category admitting a full exceptional collection such that the braid group action by mutations does not act transitively on the set of full exceptional collections up to shifts [CHS23]. Since mutations of exceptional collections do not change the generated subcategory, our example provides a surface where the braid group does not act transitively on the set of exceptional collections of maximal length.

Conventions. The term "n general points in $\mathbb{P}^2_{\mathbb{C}}$" means that there exists a nonempty Zariski open subset $U \subseteq (\mathbb{P}^2_{\mathbb{C}})^n$ such that for any $(p_1, \ldots, p_n) \in U$ [...] holds.

Acknowledgements. We thank Charles Vial for helpful discussions and explanations. We discovered the existence of the exceptional collection (2.1.2) in the context of our work in Chapter 1, where we study the transitivity of the braid group action on (numerically) exceptional collections on surfaces using a classification obtained by Vial in [Via17]. Further, we thank the anonymous referees of [Kra24a] for carefully reading our manuscript.

2.2. Exceptional Collections

We recall the basic definitions and properties of exceptional collections and semiorthogonal decompositions. For a detailed reference we refer to [Kuz14] and the references therein.

Let X be a smooth projective variety over $\mathbb{C}$ and denote by $\mathsf{D}^b(X)$ the bounded derived category of coherent sheaves on X. A *semiorthogonal decomposition* of $\mathsf{D}^b(X)$ is an ordered collection $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ of full triangulated subcategories such that

$$\mathrm{Hom}_{\mathsf{D}^b(X)}(A_i, A_j) = 0 \text{ for all } A_i \in \mathcal{A}_i, A_j \in \mathcal{A}_j, j < i$$

and the smallest triangulated subcategory of $\mathsf{D}^b(X)$ containing $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is $\mathsf{D}^b(X)$. We write

$$\mathsf{D}^b(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$$

for such a semiorthogonal decomposition. A full triangulated subcategory $\mathcal{A} \subseteq \mathsf{D}^b(X)$ is called *admissible* if the inclusion functor $\mathcal{A} \hookrightarrow \mathsf{D}^b(X)$ admits both a right and a left adjoint. Such an admissible subcategory gives rise to the semiorthogonal decompositions $\mathsf{D}^b(X) = \langle \mathcal{A}^\perp, \mathcal{A} \rangle = \langle \mathcal{A}, {}^\perp\mathcal{A} \rangle$, where

$${}^\perp\mathcal{A} := \{F \in \mathsf{D}^b(X) \mid \mathrm{Hom}_{\mathsf{D}^b(X)}(F, A) = 0 \text{ for all } A \in \mathcal{A}\}$$
$$\text{and } \mathcal{A}^\perp := \{F \in \mathsf{D}^b(X) \mid \mathrm{Hom}_{\mathsf{D}^b(X)}(A, F) = 0 \text{ for all } A \in \mathcal{A}\}$$

are the *left-* and *right-orthogonal complements* of $\mathcal{A}$. If $\mathcal{A}$ is admissible, so are ${}^\perp\mathcal{A}$ and $\mathcal{A}^\perp$.

Recall the following definitions from Section 1.2.1: An object $E \in \mathsf{D}^b(X)$ is called *exceptional* if $\mathrm{Hom}_{\mathsf{D}^b(X)}(E, E) = \mathbb{C}$ and $\mathrm{Hom}_{\mathsf{D}^b(X)}(E, E[k]) = 0$ for all $k \neq 0$. A collection $(E_1, \ldots, E_n)$ of exceptional objects is called an *exceptional collection* if

$$\mathrm{Hom}_{\mathsf{D}^b(X)}(E_i, E_j[k]) = 0 \quad \text{for all } j < i \text{ and all } k \in \mathbb{Z}.$$

The full triangulated subcategory $\langle E_1, \ldots, E_n \rangle \subseteq \mathsf{D}^b(X)$ generated by an exceptional collection $(E_1, \ldots, E_n)$ is always admissible; in particular, its left- and right-orthogonal complements are again admissible. An exceptional collection $(E_1, \dots, E_n)$ is *full* if it generates $\mathsf{D}^b(X)$, i.e., $\langle E_1, \dots, E_n \rangle = \mathsf{D}^b(X)$; equivalently $\langle E_1, \dots, E_n \rangle^{\perp} = 0 = {}^{\perp}\langle E_1, \dots, E_n \rangle$.

A semiorthogonal decomposition $\mathsf{D}^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ yields a direct sum decomposition of the Grothendieck group of $\mathsf{D}^b(X)$:

$$\mathsf{K}_0(X) = \mathsf{K}_0(\mathcal{A}_1) \oplus \cdots \oplus \mathsf{K}_0(\mathcal{A}_n).$$

An exceptional collection $(E_1, \dots, E_n)$ is of *maximal length* if there exists no further exceptional object $F \in \mathsf{D}^b(X)$ such that $(E_1, \dots, E_n, F)$ is an exceptional collection. Because $\langle E_i \rangle \cong \mathsf{D}^b(\operatorname{Spec} \mathbb{C})$ for an exceptional object E_i, we have $\mathsf{K}_0(\langle E_i \rangle) = \mathbb{Z}[E_i]$. Thus, if $\mathsf{K}_0(X)$ is finitely generated as an abelian group and $n = \operatorname{rk} \mathsf{K}_0(X)$, then any exceptional collection of length n is of maximal length.

Assume that $\mathsf{K}_0(X)$ is finitely generated and $(E_1, \dots, E_n)$ is an exceptional collection of length $n = \operatorname{rk} \mathsf{K}_0(X)$. The additivity of K_0 among semiorthogonal decompositions implies that $\mathsf{K}_0(\mathcal{A}) = \operatorname{tors}(\mathsf{K}_0(X))$ is a finite group, where $\mathcal{A} = \langle E_1, \dots, E_n \rangle^{\perp}$. If $\mathcal{A} \subseteq \mathsf{D}^b(X)$ is a nonzero admissible subcategory with finite $\mathsf{K}_0(\mathcal{A})$, then by definition $\mathcal{A}$ is a *quasi phantom* and if additionally $\mathsf{K}_0(\mathcal{A}) = 0$, then $\mathcal{A}$ is called a *phantom*.

Let $\mathcal{A} \subseteq \mathsf{D}^b(X)$ and $\mathcal{B} \subseteq \mathsf{D}^b(Y)$ be full triangulated subcategories. Then $\mathcal{A} \boxtimes \mathcal{B} \subseteq \mathsf{D}^b(X \times Y)$ denotes the smallest full triangulated subcategory of $\mathsf{D}^b(X \times Y)$ which is closed under direct summands and contains all objects of the form $\mathrm{L}p_X^* A \otimes^{\mathrm{L}} \mathrm{L}p_Y^* B$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Following [GO13, Def. 1.9] an admissible subcategory $\mathcal{A} \subseteq \mathsf{D}^b(X)$ is called a *universal phantom* if for all smooth projective varieties Y the category $\mathcal{A} \boxtimes \mathsf{D}^b(Y)$ is a phantom.

2.3. Segre–Harbourne–Gimigliano–Hirschowitz Conjecture

Let X be the blow-up of the projective plane $\mathbb{P}^2_{\mathbb{C}}$ in a set of closed points $p_1, \dots, p_n \in \mathbb{P}^2_{\mathbb{C}}$. Denote by $E_i \subseteq X$ the (-1)-curve over the point p_i and recall that $\operatorname{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_n$, where H is the pullback of a hyperplane in $\mathbb{P}^2_{\mathbb{C}}$. The class of a divisor D on X can be uniquely written as

$$D = dH - \sum_{i=1}^{n} m_i E_i$$

for some $d, m_i \in \mathbb{Z}$. Moreover, the intersection product satisfies $H^2 = 1$, $E_i^2 = -1$, $H \cdot E_i = 0$, and $E_i \cdot E_j = 0$ for all $i \neq j$. If $d > 0$ and $m_i \geq 0$, the space of global sections $H^0(X, \mathcal{O}_X(D))$ can be identified with the space of homogeneous polynomials $P \in \mathbb{C}[X, Y, Z]$ of degree d such that P vanishes to order $\geq m_i$ at p_i. If the points are chosen in general position, meaning that $h^0(D) := \dim H^0(X, \mathcal{O}_X(D))$ is minimal, then the following conjecture due to Segre–Harbourne–Gimigliano–Hirschowitz predicts the value of $h^0(D)$.

Conjecture 2.3.1 (SHGH)**.** *Let $d > 0$ and $m_i \geq 0$, $1 \leq i \leq n$, be integers. For X the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in n general points, the divisor $D := dH - \sum_{i=1}^{n} m_i E_i$ satisfies*

$$\dim H^0(X, \mathcal{O}_X(D)) = \max(0, \chi(X, \mathcal{O}_X(D)))$$

or there exists a (-1)-curve $C \subseteq X$ such that $C \cdot D \leq -2$.

Note that the generality constraints imposed on the points p_i depend on the type of divisor D, i.e., on the tuple $(d, m_1, \dots, m_n)$. If one requires $h^0(D)$ to be minimal for *all* tuples $(d, m_1, \dots, m_n)$, then the points have to be chosen very general.

A divisor $D = dH - \sum_{i=1}^{n} m_i E_i$ is said to be in *standard form* if $d > 0$, $m_i \geq 0$, $d \geq m_1 \geq \cdots \geq m_n$, and $d - m_1 - m_2 - m_3 \geq 0$. The following Lemma 2.3.2 is certainly well-known, see, e.g., [CM11, Prop. 1.4]. As it will be used in the proof of Theorem 2.1.1, we provide a proof here.

Lemma 2.3.2. *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in n points. If $D = dH - \sum_{i=1}^n m_i E_i$ is in standard form and $C \subseteq X$ is a (-1)-curve, then $D \cdot C \geq 0$.*

Proof. Let $C \subseteq X$ be a (-1)-curve. If $C = E_i$ for some i, then $D \cdot C = m_i \geq 0$. If $C \neq E_i$ for all i, then C is the strict transform of a curve in $\mathbb{P}^2_{\mathbb{C}}$, thus linearly equivalent to $eH - \sum_i f_i E_i$ with $e > 0$, and $f_i \geq 0$ for $1 \leq i \leq n$. Consider the divisors $G_1 \coloneqq H - E_1$, $G_2 \coloneqq 2H - E_1 - E_2$, and $G_j \coloneqq 3H - \sum_{i=1}^j E_i$ for $3 \leq j \leq n$. By assumption, D is a linear combination of H and G_j, $1 \leq j \leq n$, with nonnegative coefficients. The divisors H, G_1, and G_2 are nef. Further, $G_j \cdot C \geq G_n \cdot C = -K_X \cdot C$ for $3 \leq j \leq n$. Since $-K_X \cdot C = 1$, the lemma follows. □

The SHGH Conjecture is known to be true in various cases of low multiplicity. Alternatively, for a single explicit divisor D it is possible to compute the actual value of $h^0(D)$ using a computer. We will use the following known cases to show that the collection in Theorem 2.1.1 is exceptional:

Theorem 2.3.3 ([DJ07, Thm. 34], [CM11, Thm. 0.1])**.** *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in n general points and let $D = dH - \sum_{i=1}^n m_i E_i$ be a divisor with $d > 0$ and $m_i \geq 0$.*

(i) *If either all $m_i \leq 11$, or*
(ii) *if $n = 10$, $m_1 = m_2 = \cdots = m_{10}$, and $d/m_1 \geq 174/55$,*

then the SHGH Conjecture holds for D, i.e.,

$$\dim H^0(X, \mathcal{O}_X(D)) = \max(0, \chi(X, \mathcal{O}_X(D))),$$

or there exists a (-1)-curve $C \subseteq X$ such that $C \cdot D \leq -2$.

2.4. Height and Pseudoheight of Exceptional Collections

Kuznetsov introduced in [Kuz15] the so-called *height* of an exceptional collection $\langle E_1, \dots, E_n \rangle \subseteq \mathsf{D}^b(X)$: If $\mathcal{D}$ is a smooth and proper dg-category and $\mathcal{B} \subseteq \mathcal{D}$ a dg-subcategory, Kuznetsov defines the *normal Hochschild cohomology* $\mathrm{NHH}^\bullet(\mathcal{B}, \mathcal{D})$ of $\mathcal{B}$ in $\mathcal{D}$ as a certain dg-module [Kuz15, Def. 3.2]. The height of an exceptional collection $(E_1, \dots, E_n)$ is then defined as

$$\mathrm{h}(E_1, \dots, E_n) \coloneqq \min\{k \in \mathbb{Z} \mid \mathrm{NHH}^k(\mathcal{E}, \mathcal{D}) \neq 0\}$$

where $\mathcal{D}$ is a dg-enhancement of $\mathsf{D}^b(X)$ and $\mathcal{E}$ the dg-subcategory of $\mathcal{D}$ generated by the exceptional objects $(E_1, \dots, E_n)$. In general, the normal Hochschild cohomology $\mathrm{NHH}^\bullet(\mathcal{E}, \mathcal{D})$ can be computed using a spectral sequence [Kuz15, Prop. 3.7]. For our purpose it will be sufficient to consider a coarser invariant of an exceptional collection, the so-called *pseudoheight*.

Definition 2.4.1 ([Kuz15, Def. 4.4, Def. 4.9])**.** For any two objects $F, F' \in \mathsf{D}^b(X)$ define the *relative height* as

$$\mathrm{e}(F, F') \coloneqq \inf\{k \in \mathbb{Z} \mid \mathrm{Ext}^k(F, F') \neq 0\}.$$

For an exceptional collection $(E_1, \dots, E_n)$ the *pseudoheight* is

$$\mathrm{ph}(E_1, \dots, E_n) \coloneqq \min_{1 \leq a_0 < \cdots < a_p \leq n} \left(\mathrm{e}(E_{a_0}, E_{a_1}) + \cdots + \mathrm{e}(E_{a_{p-1}}, E_{a_p}) + \mathrm{e}(E_{a_p}, S^{-1}(E_{a_0})) - p \right),$$

where $S = - \otimes \omega_X[\dim X]$ is the Serre functor of $\mathsf{D}^b(X)$. The *anticanonical pseudoheight* is

$$\mathrm{ph}_{\mathrm{ac}}(E_1,\dots,E_n) \coloneqq \min_{1\le a_0<\cdots<a_p\le n} \left(\mathrm{e}(E_{a_0},E_{a_1})+\cdots+\mathrm{e}(E_{a_{p-1}},E_{a_p})+\mathrm{e}(E_{a_p},E_{a_0}\otimes\omega_X^{-1})-p\right).$$

Clearly, $\mathrm{ph}_{\mathrm{ac}} = \mathrm{ph} - \dim X$.

Lemma 2.4.2 ([Kuz15, Lem. 4.5])**.** *For an exceptional collection* $(E_1,\dots,E_n)$ *in* $\mathsf{D}^b(X)$ *we have* $\mathrm{h}(E_1,\dots,E_n) \ge \mathrm{ph}(E_1,\dots,E_n)$.

We will use the following criterion to show that the exceptional collection in Theorem 2.1.1 is not full.

Proposition 2.4.3 ([Kuz15, Prop. 6.1])**.** *Let X be a smooth projective variety and* $(E_1,\dots,E_n)$ *an exceptional collection in* $\mathsf{D}^b(X)$*. If* $\mathrm{h}(E_1,\dots,E_n) > 0$*, then* $(E_1,\dots,E_n)$ *is not full.*

In particular, if $\mathrm{ph}_{\mathrm{ac}}(E_1,\dots,E_n) > -\dim X$, then the collection is not full.

2.5. Proof of Theorem 2.1.1

Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 10 general points. Using the Beilinson collection $\mathsf{D}^b(\mathbb{P}^2_{\mathbb{C}}) = \langle \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(H), \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(2H)\rangle$, applying Orlov's blow-up formula [Orl92], and applying right mutations to the torsion sheaves, one obtains the full exceptional collection

$$\mathsf{D}^b(X) = \langle \mathcal{O}_X, \mathcal{O}_X(E_1), \dots, \mathcal{O}_X(E_{10}), \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle$$

consisting of line bundles. In particular, we obtain

$$\mathsf{K}_0(X) = \mathbb{Z}[\mathcal{O}_X] \oplus \mathbb{Z}[\mathcal{O}_X(E_1)] \oplus \cdots \oplus \mathbb{Z}[\mathcal{O}_X(E_{10})] \oplus \mathbb{Z}[\mathcal{O}_X(H)] \oplus \mathbb{Z}[\mathcal{O}_X(2H)] \cong \mathbb{Z}^{13}.$$

Recall from Section 1.5 that $\mathrm{Pic}(X)$ admits an orthogonal decomposition $\mathrm{Pic}(X) = K_X^{\perp} \oplus \mathbb{Z}K_X$ and an involution $\iota\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ given by $\iota \coloneqq -\mathrm{id}_{K_X^{\perp}} \oplus \mathrm{id}_{\mathbb{Z}K_X}$. We compute

$$D_i \coloneqq \iota(E_i) = -6H + 2\sum_{j=1}^{10} E_j - E_i \quad \text{and} \quad F \coloneqq \iota(H) = -19H + 6\sum_{i=1}^{10} E_i.$$

In particular, ι fixes the canonical class and thus, by Lemma 1.2.11,

$$(\mathcal{O}_X, \mathcal{O}_X(D_1), \dots, \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F)) \tag{2.5.1}$$

is a *numerically exceptional collection*, i.e., it is semiorthogonal with respect to the Euler pairing

$$\chi(F,G) \coloneqq \sum_{i\in\mathbb{Z}} (-1)^i \dim \mathrm{Hom}_{\mathsf{D}^b(X)}(F, G[i])$$

and each object F in the collection satisfies $\chi(F,F) = 1$. Moreover, it is clear that the image of (2.5.1) is a basis of the Grothendieck group $\mathsf{K}_0(X) \cong \mathbb{Z}^{13}$, thus the collection is of maximal length.

Proof of the Theorem 2.1.1. We first verify that the collection (2.5.1) is exceptional. Since the collection is numerically exceptional and consists of sheaves, it suffices to check the vanishing of Hom- and Ext^2-spaces. Via Serre duality, the computation of an Ext^2-space can be done by computing global sections of a divisor. Thus, abbreviating

$\hom(-,-) = \dim \operatorname{Hom}(-,-)$ and $\operatorname{ext}^k(-,-) = \dim \operatorname{Ext}^k(-,-)$, we have to show that the following dimensions are zero:

$$\begin{aligned}
\hom(\mathcal{O}_X(2F), \mathcal{O}_X(F)) &= h^0(-F),\\
\operatorname{ext}^2(\mathcal{O}_X(2F), \mathcal{O}_X(F)) &= h^2(-F) = h^0(K_X + F),\\
\hom(\mathcal{O}_X(2F), \mathcal{O}_X(D_i)) &= h^0(D_i - 2F),\\
\operatorname{ext}^2(\mathcal{O}_X(2F), \mathcal{O}_X(D_i)) &= h^2(D_i - 2F) = h^0(K_X - D_i + 2F),\\
\hom(\mathcal{O}_X(2F), \mathcal{O}_X) &= h^0(-2F),\\
\operatorname{ext}^2(\mathcal{O}_X(2F), \mathcal{O}_X) &= h^2(-2F) = h^0(K_X + 2F),\\
\hom(\mathcal{O}_X(F), \mathcal{O}_X(D_i)) &= h^0(D_i - F),\\
\operatorname{ext}^2(\mathcal{O}_X(F), \mathcal{O}_X(D_i)) &= h^2(D_i - F) = h^0(K_X - D_i + F),\\
\hom(\mathcal{O}_X(F), \mathcal{O}_X) &= h^0(-F),\\
\operatorname{ext}^2(\mathcal{O}_X(F), \mathcal{O}_X) &= h^2(-F) = h^0(K_X + F),\\
\hom(\mathcal{O}_X(D_i), \mathcal{O}_X(D_j)) &= h^0(D_j - D_i),\\
\operatorname{ext}^2(\mathcal{O}_X(D_i), \mathcal{O}_X(D_j)) &= h^2(D_j - D_i) = h^0(K_X - D_j + D_i),\\
\hom(\mathcal{O}_X(D_i), \mathcal{O}_X) &= h^0(-D_i),\\
\operatorname{ext}^2(\mathcal{O}_X(D_i), \mathcal{O}_X) &= h^2(-D_i) = h^0(K_X + D_i),
\end{aligned}$$

where $1 \le i, j \le 10$, $i \ne j$. The vanishing holds trivially if the divisor has negative intersection with H, is of the form $D_j - D_i = E_i - E_j$, or is of the form $K_X - D_j + D_i = K_X - E_i + E_j$. The remaining cases are

$$\text{(2.5.2)} \quad -F = 19H - 6\sum_{j=1}^{10} E_j, \quad -2F = 38H - 12\sum_{j=1}^{10} E_j, \quad -D_i = 6H - 2\sum_{j=1}^{10} E_j + E_i,$$
$$D_i - F = 13H - 4\sum_{j=1}^{10} E_j - E_i, \quad D_i - 2F = 32H - 10\sum_{j=1}^{10} E_j - E_i.$$

Up to permutation of the points, these divisors are in standard form. Thus by Lemma 2.3.2, if D is one of the divisors in (2.5.2), then $C \cdot D \ge 0$ holds for any (-1)-curve $C \subseteq X$. If $D \ne -2F$, then the multiplicities of D are bounded by 11, thus $h^0(D) = \chi(D) = 0$ by Theorem 2.3.3 (i). If $D = -2F$, then we compute $38/12 \ge 174/55$. Hence, $h^0(-2F) = \chi(-2F) = 0$ by Theorem 2.3.3 (ii). Therefore, (2.5.1) is exceptional.

To show that (2.5.1) is not full, by Proposition 2.4.3 and Lemma 2.4.2 it suffices to show that the anticanonical pseudoheight $\mathrm{ph}_{\mathrm{ac}}$ of (2.5.1) is at least -1. In the following, we show that $\mathrm{ph}_{\mathrm{ac}} \ge 0$. Recall that

$$\text{(2.5.3)} \quad \mathrm{ph}_{\mathrm{ac}} = \min_{1 \le a_0 < \cdots < a_p \le 13} \left(\mathrm{e}(E_{a_0}, E_{a_1}) + \cdots + \mathrm{e}(E_{a_{p-1}}, E_{a_p}) + \mathrm{e}(E_{a_p}, E_{a_0} \otimes \omega_X^{-1}) - p \right),$$

where the E_{a_i} are the exceptional objects in (2.5.1). Since (2.5.1) consists of sheaves, $\mathrm{e}(E_{a_i}, E_{a_{i+1}})$ and $\mathrm{e}(E_{a_p}, E_{a_0} \otimes \omega_X^{-1})$ take values in $\{0, 1, 2, \infty\}$. First, if $p = 0$, then the expression under the minimum in (2.5.3) is $\mathrm{e}(E_{a_0}, E_{a_0} \otimes \omega_X^{-1}) \ge 0$. Next, if $p \ge 1$ and if we know that $\mathrm{e}(E_{a_i}, E_{a_{i+1}}) \ge 1$ for all $0 \le i \le p-1$, then the expression under the minimum in (2.5.3) is greater or equal than $\mathrm{e}(E_{a_p}, E_{a_0} \otimes \omega_X^{-1}) \ge 0$. Hence, it is enough to show that the following dimensions vanish:

$$\begin{aligned}
\hom(\mathcal{O}_X, \mathcal{O}_X(D_i)) &= h^0(D_i),\\
\hom(\mathcal{O}_X, \mathcal{O}_X(F)) &= h^0(F),\\
\hom(\mathcal{O}_X, \mathcal{O}_X(2F)) &= h^0(2F),\\
\hom(\mathcal{O}_X(D_i), \mathcal{O}_X(D_j)) &= h^0(D_j - D_i),\\
\hom(\mathcal{O}_X(D_i), \mathcal{O}_X(F)) &= h^0(F - D_i),\\
\hom(\mathcal{O}_X(D_i), \mathcal{O}_X(2F)) &= h^0(2F - D_i),\\
\hom(\mathcal{O}_X(F), \mathcal{O}_X(2F)) &= h^0(F),
\end{aligned}$$

where $1 \le i, j \le 10$ and $i \ne j$. All these divisors have either negative intersection with H or are of the form $D_j - D_i = E_i - E_j$, thus the vanishing holds for trivial reasons. Hence, $\mathrm{ph}_{\mathrm{ac}} \ge 0$ and we conclude that (2.5.1) is not full. □

Corollary 2.5.4. *The admissible subcategory*

$$\mathcal{A} = \langle \mathcal{O}_X, \mathcal{O}_X(D_1), \dots, \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F)\rangle^{\perp}$$

is a universal phantom subcategory of $\mathsf{D}^b(X)$.

Proof. The Chow motive of X with integer coefficients is of Lefschetz type and $\mathsf{K}_0(\mathcal{A}) = 0$. By [GO13, Cor. 4.3] the K-motive of $\mathcal{A}$ with integer coefficients vanishes and [GO13, Prop. 4.4] shows that $\mathcal{A}$ is a universal phantom. □

Remark 2.5.5 (On the base field)*.* Theorem 2.1.1 is stated using the base field $\mathbb{C}$, since we rely on [CM11, Thm. 0.1] in Theorem 2.3.3. Replacing the usage of [CM11, Thm. 0.1] by a computer aided computation as Proposition 2.A.2, it is possible to deduce that the conclusion of Theorem 2.1.1 also holds over an algebraically closed field of characteristic zero.

Remark 2.5.6 (Hochschild cohomology of the phantom)*.* Using the Hochschild–Kostant–Rosenberg isomorphism, we compute in Lemma 2.B.5 that the Hochschild cohomology of X satisfies:

$$\dim \mathrm{HH}^0(X) = 1, \quad \dim \mathrm{HH}^1(X) = 0, \quad \dim \mathrm{HH}^2(X) = 12,$$
$$\text{and} \quad \dim \mathrm{HH}^i(X) = 0 \quad \text{for } i \ge 3.$$

Applying the techniques from [Kuz15] we further compute in Propositions 2.B.4 and 2.B.9 that the height of (2.5.1) is 4 and the Hochschild cohomology of $\mathcal{A}$ has the following dimensions:

$$\begin{aligned}
&\dim \mathrm{HH}^0(\mathcal{A}) = 1, && \dim \mathrm{HH}^1(\mathcal{A}) = 0, && \dim \mathrm{HH}^2(\mathcal{A}) = 12, && \dim \mathrm{HH}^3(\mathcal{A}) = 446,\\
&\dim \mathrm{HH}^4(\mathcal{A}) = 853, && \dim \mathrm{HH}^5(\mathcal{A}) = 420, && \dim \mathrm{HH}^i(\mathcal{A}) = 0 \text{ for } i \ge 6.
\end{aligned}$$

In particular, the restriction morphism $\mathrm{HH}^i(X) \to \mathrm{HH}^i(\mathcal{A})$ is an isomorphism for $0 \le i \le 2$ and a monomorphism for $i = 3$. As explained in [Kuz15, Prop. 4.8], this implies that the formal deformation spaces of $\mathsf{D}^b(X)$ and $\mathcal{A}$ are isomorphic.

Appendices to Chapter 2

2.A. Verifying the SHGH Conjecture in Explicit Cases

Let K be field and let $p_1, \ldots, p_n \in \mathbb{P}^2_K(K)$ be K-valued points. Let $\pi\colon X \to \mathbb{P}^2_K$ be the blow-up in $p_1, \ldots, p_n$ and recall that $\operatorname{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_n$, where E_i is the exceptional divisor over the point p_i and H the pullback of the class of a hyperplane in $\mathbb{P}^2_K$. For integers $d, m_1, \ldots m_n \geq 0$ consider the divisor $D := dH - \sum_{i=1}^n m_i E_i$. In this section we explain how to compute the dimension $h^0(D) = \dim H^0(X, \mathcal{O}_X(D))$ for a field K of characteristic zero and points $p_1, \ldots, p_n$ in sufficiently general position, meaning that $h^0(D)$ is minimal. This method of computation is later used in Proposition 2.B.3 and can also be used to verify the computations in Theorem 2.1.1 independently from the literature. In particular, this method can be used, as outlined in Remark 2.5.5, to prove the conclusion of Theorem 2.1.1 over an algebraically closed field of characteristic zero.

To begin with, we recall how global sections of $\mathcal{O}_X(D)$ can be interpreted as explicit polynomials: First, note that

$$\pi_* \mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^2_K}(dH) \otimes \mathcal{I}_{p_1}^{m_1} \otimes \cdots \otimes \mathcal{I}_{p_n}^{m_n},$$

where $\mathcal{I}_{p_i}$ is the ideal sheaf of the point p_i. Thus, by taking global sections, we can identify $H^0(X, \mathcal{O}_X(D))$ with homogeneous polynomials $P \in K[X, Y, Z]_{\deg d}$ of degree d, which vanish at p_i to order $\geq m_i$ for $1 \leq i \leq n$.

Recall that if a point $p \in \mathbb{P}^2_K$ lies in the affine open subset $\{Z \neq 0\} \subseteq \mathbb{P}^2_K$ and if $\operatorname{char} K = 0$ or $\operatorname{char} K > \deg P = d$, then P vanishes to order $\geq m$ at $p = [p_1 : p_2 : 1] \in \{Z \neq 0\}$ if and only if

$$\frac{\partial^{i+j} P(X, Y, 1)}{\partial^i X \partial^j Y}(p_1, p_2, 1) = 0 \text{ for all } 0 \leq i + j < m.$$

For simplicity assume that all p_i lie in the affine open subset $\{Z \neq 0\} \subseteq \mathbb{P}^2_K$ and write $[p_{i1} : p_{i2} : 1] = p_i \in \{Z \neq 0\} \subseteq \mathbb{P}^2_K$. Then $H^0(X, \mathcal{O}_X(D))$ can be identified with the kernel of the K-linear map

$$K[X, Y, Z]_{\deg d} \ni P \mapsto \left(\frac{\partial^{i+j} P(X, Y, 1)}{\partial^i X \partial^j Y}(p_{l1}, p_{l2}, 1) \right)_{i,j,l} \in K^N, \tag{2.A.1}$$

for $N = \sum_{l=1}^n \binom{m_l+1}{2}$ (if $\operatorname{char} K = 0$ or $\operatorname{char} K > d$). This shows that the inequality

$$h^0(D) \geq \max \left(0, \chi(D) = \binom{d+2}{2} - \sum_{l=1}^n \binom{m_l + 1}{2} \right)$$

is always fulfilled and is an equality if and only if (2.A.1) has maximal rank

$$\min \left(\sum_{l=1}^n \binom{m_l + 1}{2}, \binom{d+2}{2} \right).$$

Moreover, the map

$$(\mathbb{P}^2_K)^n \ni (p_1, \ldots, p_n) \mapsto h^0(D)$$

is upper semi-continuous and if the points $p_l = [p_{l1} : p_{l2} : 1]$ have integer coefficients $p_{l1}, p_{l2} \in \mathbb{Z}$, then the linear map (2.A.1) is defined over $\mathbb{Z}$. Thus, the following implication holds: If $H^0(X, \mathcal{O}_X(D)) = \max(0, \chi(D))$ holds for some finite K with $\operatorname{char} K > H \cdot D$ and K-valued points $p_0, \ldots, p_n$, then $H^0(X, \mathcal{O}_X(D)) = \max(0, \chi(D))$ holds for a field K of characteristic zero and any set $p_1, \ldots, p_n$ of points in general position. Hence, we can verify the equality $H^0(X, \mathcal{O}_X(D)) = \max(0, \chi(D))$ by a computation over a finite field.

The following procedure can be executed in Singular [Dec+21] and is a modified version of the code used in [LU03], available at [Laf].

```
ring R=32003,(x,y,z),dp;
LIB "general.lib";

proc comp_eff_vir (int d,int mo,int m,int n)
{
 int i,eff,virt,h;
 intvec hi;
 ideal I = 1;
 for(i=1; i<=n;i=i+1)
 {
  int p(i) = random(-100,100);
  int q(i) = random(-100,100);
  ideal K(i) = x-p(i)*z,y-q(i)*z;
  ideal I(i) = std(K(i)^(m));
  I = std(intersect(I,I(i)));
 }

 int po = random(-100,100);
 int qo = random(-100,100);
 ideal Ko = x-po*z,y-qo*z;
 ideal Io = std((Ko)^(mo));
 I = std(intersect(I,Io));
 hi = hilb(I,2);

 h = min(d+1, size(hi));
 eff = (((d+1)*(d+2)) div 2 )-sum(hi,1..h);
 virt = (((d+1)*(d+2)) div 2)-((mo*(mo+1)) div 2)-((n*(m*(m+1))) div 2);

 return (eff,virt);
}
```

Proposition 2.A.2. *Let K be an algebraically closed field of characteristic zero. Let X be the blow-up of $\mathbb{P}^2_K$ in $n+1$ general points $p_0, \ldots p_n \in \mathbb{P}^2_K$ and let*

$$D = dH - m_0 E_0 - m \sum_{i=1}^{n} E_i \quad \textit{for} \quad 0 < d, m_0, m < 32003.$$

If the return value `(eff,virt)` *of the procedure* `comp_eff_vir(d,`m_0`,`m`,`n`)` *satisfies* $\texttt{eff} = \max(0, \texttt{virt})$*, then* $h^0(D) = \texttt{eff} = \max(0, \texttt{virt})$*.*

Proof. The procedure `comp_eff_vir(d,mo,m,n)` takes 4 integers $d, m_0, m, n \geq 0$ and returns the actual dimension $h^0(D)$ for the blow-up of $n+1$ random points $p_0, \ldots, p_n$ and divisor $D = dH - m_0 E_0 - m \sum_{j=1}^{n} E_i$ as the first value `eff`, as well as $\chi(D)$ as the second value `virt`. The whole computation is performed with coefficients in the field $\mathbb{F}_{32003}$. Thus, by the upper semi-continuity of the map

$$(\mathbb{P}^2_K)^{n+1} \ni (p_0, \ldots, p_n) \mapsto h^0(D)$$

and the lower bound $h^0(D) \geq \max(0, \chi(D))$, the statement of the proposition follows.

In the following we describe the procedure `comp_eff_vir()` in words: In the first step, a set of random points $[p_i : q_i : 1] \in \{Z \neq 0\} \subseteq \mathbb{P}^2_{\mathbb{F}_{32003}}$ is chosen. These correspond to the maximal ideals

$$(X - p_i Z, Y - q_i Z, q_i X - p_i Y) = (X - p_i Z, Y - q_i Z) \subseteq \mathbb{F}_{32003}[X, Y, Z].$$

In the second step, the second Hilbert series of $\mathbb{F}_{32003}[X, Y, Z]/I$ is computed, where

$$I = (X - p_0 Z, Y - q_0 Z)^{m_0} \cap (X - p_1 Z, Y - q_1 Z)^m \cap \cdots \cap (X - p_n Z, Y - q_n Z)^m.$$

Recall that the second Hilbert series of R/I are the coefficients of the polynomial Q satisfying

$$\sum_{k=0}^{\infty} \dim((R/I)_{\deg k}) t^k = \frac{Q(t)}{(1-t)^{\dim R/I}} \tag{2.A.3}$$

where $\dim R/I$ is the affine Krull dimension. In our case $\dim R/I = 1$ and solving (2.A.3) recursively yields the formula for the actual dimension of $h^0(D)$ used to determine the variable `eff` in `comp_eff_vir()`. □

2.B. Height and Hochschild Cohomology

Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 10 general points. In Theorem 2.1.1 we showed that the line bundles

$$(\mathcal{O}_X, \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F)), \tag{2.B.1}$$

$$\text{where} \quad D_i := -6H + 2\sum_{j=1}^{10} E_j - E_i \quad \text{and} \quad F := -19H + 6\sum_{i=1}^{10} E_i,$$

form an exceptional collection in $\mathsf{D}^b(X)$. In this section we compute the anticanonical pseudoheight of (2.B.1) and show that its height is equal to its pseudoheight; see Proposition 2.B.4. After that we compute the Hochschild cohomology of

$$\mathcal{A} = \langle \mathcal{O}_X, \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F) \rangle^{\perp} \subseteq \mathsf{D}^b(X);$$

see Proposition 2.B.9.

Lemma 2.B.2. *All the morphisms in (2.B.1) are centered in* Ext^2*, meaning that* $\operatorname{Ext}^k(E_i, E_j) = 0$ *for all* $k \neq 2$*,* $i \neq j$*, and* E_i, E_j *in (2.B.1). Moreover, the objects* $\mathcal{O}_X(D_i)$ *and* $\mathcal{O}_X(D_j)$ *are orthogonal for* $i \neq j$*, i.e.,* $\mathrm{e}(\mathcal{O}_X(D_i), \mathcal{O}_X(D_j)) = \infty$*.*

Proof. Since $H^\bullet(X, \mathcal{O}_X(E_j - E_i)) = 0$, $\mathcal{O}_X(D_i)$ and $\mathcal{O}_X(D_j)$ are orthogonal. We have already seen in the proof of Theorem 2.1.1 that (2.B.1) admits no Hom-spaces between any pair of distinct objects. In order to conclude that there are also no Ext^1-spaces it suffices to show that $\mathrm{ext}^2(E_i, E_j) = \chi(E_i, E_j)$ for all $i \neq j$. We verify:

$$\begin{aligned}
\mathrm{ext}^2(\mathcal{O}_X, \mathcal{O}_X(D_i)) &= h^2(D_i) = h^0(K_X - D_i) = 1, \\
\mathrm{ext}^2(\mathcal{O}_X, \mathcal{O}_X(F)) &= h^2(F) = h^0(K_X - F) = 3, \\
\mathrm{ext}^2(\mathcal{O}_X, \mathcal{O}_X(2F)) &= h^2(2F) = h^0(K_X - 2F) = 6, \\
\mathrm{ext}^2(\mathcal{O}_X(D_i), \mathcal{O}_X(D_j)) &= h^2(D_j - D_i) = h^0(K_X - E_i + E_j) = 0, \\
\mathrm{ext}^2(\mathcal{O}_X(D_i), \mathcal{O}_X(F)) &= h^2(F - D_i) = h^0(K_X - F + D_i) = 2, \\
\mathrm{ext}^2(\mathcal{O}_X(D_i), \mathcal{O}_X(2F)) &= h^2(2F - D_i) = h^0(K_X - 2F + D_i) = 5, \\
\mathrm{ext}^2(\mathcal{O}_X(F), \mathcal{O}_X(2F)) &= h^2(F) = h^0(K_X - F) = 3.
\end{aligned}$$

Again all the divisors are in standard form and have multiplicities bounded by 11 and thus belong to the known cases of the SHGH Conjecture [DJ07]. □

Proposition 2.B.3. *The anticanonical pseudoheight of* (2.B.1) *is* 2.

Proof. Since $\mathrm{e}(\mathcal{O}_X(D_i), \mathcal{O}_X(D_j)) = \infty$, the anticanonical pseudoheight is the minimum of the following numbers:

$$\begin{aligned}
&\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(-K_X)), \\
&\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(D_i)) + \mathrm{e}(\mathcal{O}_X(D_i), \mathcal{O}_X(-K_X)) - 1, \\
&\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(F)) + \mathrm{e}(\mathcal{O}_X(F), \mathcal{O}_X(-K_X)) - 1, \\
&\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(2F)) + \mathrm{e}(\mathcal{O}_X(2F), \mathcal{O}_X(-K_X)) - 1, \\
&\mathrm{e}(\mathcal{O}_X(D_i), \mathcal{O}_X(F)) + \mathrm{e}(\mathcal{O}_X(F), \mathcal{O}_X(D_i - K_X)) - 1, \\
&\mathrm{e}(\mathcal{O}_X(D_i), \mathcal{O}_X(2F)) + \mathrm{e}(\mathcal{O}_X(2F), \mathcal{O}_X(D_i - K_X)) - 1, \\
&\mathrm{e}(\mathcal{O}_X(F), \mathcal{O}_X(2F)) + \mathrm{e}(\mathcal{O}_X(2F), \mathcal{O}_X(F - K_X)) - 1, \\
&\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(D_i)) + \mathrm{e}(\mathcal{O}_X(D_i), \mathcal{O}_X(F)) + \mathrm{e}(\mathcal{O}_X(F), \mathcal{O}_X(-K_X)) - 2, \\
&\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(D_i)) + \mathrm{e}(\mathcal{O}_X(D_i), \mathcal{O}_X(2F)) + \mathrm{e}(\mathcal{O}_X(2F), \mathcal{O}_X(-K_X)) - 2, \\
&\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(F)) + \mathrm{e}(\mathcal{O}_X(F), \mathcal{O}_X(2F)) + \mathrm{e}(\mathcal{O}_X(2F), \mathcal{O}_X(-K_X)) - 2, \\
&\mathrm{e}(\mathcal{O}_X(D_i), \mathcal{O}_X(F)) + \mathrm{e}(\mathcal{O}_X(F), \mathcal{O}_X(2F)) + \mathrm{e}(\mathcal{O}_X(2F), \mathcal{O}_X(D_i - K_X)) - 2, \\
&\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(D_i)) + \mathrm{e}(\mathcal{O}_X(D_i, \mathcal{O}_X(F)) + \mathrm{e}(\mathcal{O}_X(F), \mathcal{O}_X(2F)) + \mathrm{e}(\mathcal{O}_X(2F), \mathcal{O}_X(-K_X)) - 3.
\end{aligned}$$

We compute:

$$\begin{aligned}
\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(-K_X)) &= \infty, \\
\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(-K_X - D_i)) &= 1, \\
\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(-K_X - F)) &= 1, \\
\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(-K_X - 2F)) &= 1, \\
\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(D_i - K_X - F)) &= 1, \\
\mathrm{e}(\mathcal{O}_X, \mathcal{O}_X(D_i - K_X - 2F)) &= 1.
\end{aligned}$$

Note that the divisor $D_i - K_X - 2F$ does not belong to the known cases of the SHGH Conjecture. We verified the vanishing of global sections with a computer using Proposition 2.A.2. Thus, the anticanonical pseudoheight is 2. □

Proposition 2.B.4. *The height of* (2.B.1) *is* 4.

Proof. It follows from Proposition 2.B.3 that the pseudoheight is 4, thus it suffice to show that the height is equal to the pseudoheight. First, note that

$$\begin{aligned} &\mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_X(-K_X - D_i)) = \mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_X(-K_X - F)) = \mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_X(-K_X - 2F)) \\ = &\ \mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_X(D_i - K_X - F)) = \mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_X(D_i - K_X - 2F)) = 0. \end{aligned}$$

Indeed, after applying Serre duality one can show the vanishing by computing global sections of divisors which all have negative intersection with the hyperplane class H. Recall the following techniques from [Kuz15]: As before let $\mathcal{E}$ be the dg-subcategory of a dg-enhancement of $\mathsf{D}^b(X)$ generated by the exceptional collection (2.B.1). In [Kuz15, Prop. 3.7] the existence of a spectral sequence

$$E_1^{-p,q} \Rightarrow \mathrm{NHH}^{q-p}(\mathcal{E}, X)$$

is established. The E_1-page of that spectral sequence is is given by

$$E_1^{-p,q} = \bigoplus_{\substack{1 \le a_0 < \cdots < a_p \le n \\ k_0 + \cdots + k_p = q}} \mathrm{Ext}^{k_0}(E_{a_0}, E_{a_1}) \otimes \cdots \otimes \mathrm{Ext}^{k_{p-1}}(E_{a_{p-1}}, E_{a_p}) \otimes \mathrm{Ext}^{k_p}(E_{a_p}, S^{-1}(E_{a_0}))$$

with S the Serre functor of $\mathsf{D}^b(X)$. The differential on the E_1-page is given by the Yoneda-multiplication of adjacent Ext-spaces. Concerning the collection (2.B.1), there are only Ext^2-spaces between objects belonging to the exceptional collection and the term $\mathrm{Ext}^{k_p}(E_{a_p}, S^{-1}(E_{a_0})) = \mathrm{Ext}^{k_p - 2}(E_{a_p}, E_{a_0} \otimes \omega_X^{-1})$ takes nontrivial values only if $k_p = 3$. We conclude from $\dim X = 2$ that the differential on the E_1-page is zero. This shows that the height of (2.B.1) is equal to the pseudoheight. □

Lemma 2.B.5. *The Hochschild cohomology of X is given by:*

$$\mathrm{HH}^0(X) = \mathbb{C}, \quad \mathrm{HH}^1(X) = 0, \quad \mathrm{HH}^2(X) = \mathbb{C}^{12}, \quad \textit{and} \quad \mathrm{HH}^i(X) = 0 \quad \textit{for } i \ge 3.$$

Proof. The Hochschild cohomology of a smooth projective variety Y can be computed via the Hochschild–Kostant–Rosenberg isomorphism

$$\mathrm{HH}^k(Y) = \bigoplus_{p+q=k} H^q\left(Y, \bigwedge^p T_Y\right),$$

see [Swa96, Cor. 2.6]. Thus, in order to determine $\mathrm{HH}^\bullet(X)$ we have to compute $H^\bullet(X, T_X)$. Recall that for any smooth projective surface Y there is a non-degenerate pairing

$$\Omega_Y^1 \otimes \Omega_Y^1 \to \omega_Y,$$

which identifies $T_Y = \mathcal{H}om(\Omega_Y^1, \mathcal{O}_Y) \cong \Omega_Y^1 \otimes \omega_Y^{-1}$. Fix the following notation for the blow-up:

$$\begin{array}{ccc} X & \xrightarrow{\quad\pi\quad} & \mathbb{P}^2_{\mathbb{C}} \\ {\scriptstyle j}\uparrow & & \uparrow \\ E = \coprod_{i=1}^{10} E_i & \longrightarrow & Z := \{p_1, \ldots, p_{10}\}. \end{array}$$

The cotangent bundle Ω_X^1 fits into an exact sequence

$$0 \to \pi^* \Omega^1_{\mathbb{P}^2_{\mathbb{C}}} \to \Omega^1_X \to \Omega^1_{X/\mathbb{P}^2_{\mathbb{C}}} \to 0, \tag{2.B.6}$$

where the relative cotangent sheaf $\Omega^1_{X/\mathbb{P}^2_{\mathbb{C}}}$ can be identified with $j_* \Omega^1_{E/Z} = j_*\left(\bigoplus_{i=1}^{10} \Omega^1_{E_i}\right)$. Twisting (2.B.6) with $\omega_X^{-1} = \mathcal{O}_X(3H - \sum_i E_i)$, using the projection formula, and the isomorphism $T_Y \cong \Omega^1_Y \otimes \omega_Y^{-1}$ for $Y = \mathbb{P}^2_{\mathbb{C}}$ and $Y = X$, we obtain an exact sequence

$$0 \to (\pi^* T_{\mathbb{P}^2_{\mathbb{C}}}) \otimes \mathcal{O}_X\left(-\sum_{i=1}^{10} E_i\right) \to T_X \to \left(j_* \bigoplus_{i=1}^{10} \mathcal{O}_{E_i}(-2)\right) \otimes \omega_X^{-1} \to 0. \tag{2.B.7}$$

The projection formula yields

$$\pi_*\left((\pi^* T_{\mathbb{P}^2_{\mathbb{C}}}) \otimes \mathcal{O}_X\left(-\sum_{i=1}^{10} E_i\right)\right) = T_{\mathbb{P}^2_{\mathbb{C}}} \otimes \mathcal{I}_{p_1} \otimes \cdots \otimes \mathcal{I}_{p_{10}},$$

where $\mathcal{I}_{p_i}$ is the ideal sheaf of the point $p_i \in \mathbb{P}^2_{\mathbb{C}}$. Therefore we can identify global sections of $(\pi^* T_{\mathbb{P}^2_{\mathbb{C}}}) \otimes \mathcal{O}_X(-\sum_i E_i)$ with global sections of $T_{\mathbb{P}^2_{\mathbb{C}}}$ vanishing at the points p_i.

Moreover, the sheaf $(j_* \bigoplus_i \mathcal{O}_{E_i}(-2)) \otimes \omega_X^{-1}$ is supported on the exceptional divisor E and can be identified with $j_* \bigoplus_i \mathcal{O}_{E_i}(-1)$. In particular, $j_* \bigoplus_i \mathcal{O}_{E_i}(-1)$ has no global sections. Thus, applying $H^0(X, -)$ to (2.B.7) identifies $H^0(X, T_X)$ with global sections of $T_{\mathbb{P}^2_{\mathbb{C}}}$ vanishing at the points p_i.

With the help of the Euler sequence it is easy to compute that

$$H^i(\mathbb{P}^2_{\mathbb{C}}, T_{\mathbb{P}^2_{\mathbb{C}}}) = \begin{cases} \mathbb{C}^8 & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

In order to conclude that $H^0(X, T_X) = 0$, it is enough to show that if $S \to \mathbb{P}^2_{\mathbb{C}}$ is the blow-up of 4 general points, i.e., a del Pezzo surface of degree 5, then $H^0(S, T_S) = 0$. It is known, see, e.g., [DI09, § 6.3], that the automorphism group $\operatorname{Aut}(S)$ is finite. Since $\dim T_{\mathrm{id}_S} \operatorname{Aut}(S) = \dim H^0(S, T_S)$, see, e.g., [Kol96, Ex. 2.16.4], this shows $H^0(S, T_S) = 0$ and thus $H^0(X, T_X) = 0$.

In order to determine $H^i(X, T_X)$ for $i > 0$, we first dualize (2.B.6) to obtain

$$0 \to \mathcal{H}om(j_* \Omega^1_{E/Z}, \mathcal{O}_X) \to T_X \to \pi^* T_{\mathbb{P}^2_{\mathbb{C}}} \to \mathcal{E}xt^1(j_* \Omega^1_{E/Z}, \mathcal{O}_X) \to 0. \tag{2.B.8}$$

Using the identification $j_* \Omega^1_{E/Z} = j_* \bigoplus_i \mathcal{O}_{E_i}(-2) = j_* \mathcal{O}_E(2E)$, the twisted ideal sheaf sequence of E yields a locally free resolution

$$0 \to \mathcal{O}_X(E) \to \mathcal{O}_X(2E) \to j_* \mathcal{O}_E(2E) \to 0.$$

Applying $\mathcal{H}om(-, \mathcal{O}_X)$, we obtain

$$\mathcal{H}om(j_* \Omega^1_{E/Z}, \mathcal{O}_X) = \ker(\mathcal{O}_X(-2E) \to \mathcal{O}_X(-E)) \text{ and}$$
$$\mathcal{E}xt^1(j_* \Omega^1_{E/Z}, \mathcal{O}_X) = \operatorname{coker}(\mathcal{O}_X(-2E) \to \mathcal{O}_X(-E)).$$

Thus, $\mathcal{H}om(j_* \Omega^1_{E/Z}, \mathcal{O}_X) = 0$ and $\mathcal{E}xt^1(j_* \Omega^1_{E/Z}, \mathcal{O}_X) \cong j_* \bigoplus_i \mathcal{O}_{E_i}(1)$ and we obtain a short exact sequence

$$0 \to T_X \to \pi^* T_{\mathbb{P}^2_{\mathbb{C}}} \to j_* \bigoplus_{i=1}^{10} \mathcal{O}_{E_i}(1) \to 0.$$

(This sequence can also be obtained from (2.B.8) by using the right adjoint $j^!$ of j_* and computing $j^! \mathcal{O}_X = j^* \mathcal{O}_X(E)[-1]$.) Using the associated long exact sequence in cohomology and the identification $\pi_* \pi^* T_{\mathbb{P}^2_{\mathbb{C}}} = T_{\mathbb{P}^2_{\mathbb{C}}}$, we obtain

$$h^1(X, T_X) = h^0\left(E, \bigoplus_{i=1}^{10} \mathcal{O}_{E_i}(1)\right) - h^0(\mathbb{P}^2_{\mathbb{C}}, T_{\mathbb{P}^2_{\mathbb{C}}}) = 20 - 8 = 12$$

and $H^i(X, T_X) = 0$ for $i > 1$.

Finally, using $\bigwedge^2 T_X = \omega_X^{-1}$, we obtain using the Hochschild–Kostant–Rosenberg isomorphism

$$\begin{aligned} \mathrm{HH}^0(X) &= H^0(X, \mathcal{O}_X) = \mathbb{C}, \\ \mathrm{HH}^1(X) &= H^0(X, T_X) = 0, \\ \mathrm{HH}^2(X) &= H^1(X, T_X) = \mathbb{C}^{12}, \text{ and} \\ \mathrm{HH}^i(X) &= 0 \text{ for } i \geq 3. \end{aligned}$$

□

Proposition 2.B.9. *The Hochschild cohomology of*

$$\mathcal{A} = \langle \mathcal{O}_X, \mathcal{O}_X(D_1), \dots, \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F)\rangle^{\perp} \subseteq \mathsf{D}^b(X)$$

has the following dimensions:

$$\begin{aligned} &\dim \mathrm{HH}^0(\mathcal{A}) = 1, && \dim \mathrm{HH}^1(\mathcal{A}) = 0, && \dim \mathrm{HH}^2(\mathcal{A}) = 12, && \dim \mathrm{HH}^3(\mathcal{A}) = 446, \\ &\dim \mathrm{HH}^4(\mathcal{A}) = 853, && \dim \mathrm{HH}^5(\mathcal{A}) = 420, && \dim \mathrm{HH}^i(\mathcal{A}) = 0 \ \textit{for } i \geq 6. \end{aligned}$$

Proof. The Hochschild cohomology of $\mathcal{A}$ can be computed using the exact triangle

$$\mathrm{NHH}^\bullet(\mathcal{E}, X) \to \mathrm{HH}^\bullet(X) \to \mathrm{HH}^\bullet(\mathcal{A}), \tag{2.B.10}$$

established in [Kuz15, Thm. 3.3]. We already determined $\mathrm{HH}^\bullet(X)$ in Lemma 2.B.5. To compute the dimensions of $\mathrm{NHH}^\bullet(\mathcal{E}, X)$, we will use the spectral sequence from [Kuz15, Prop. 3.7] as in the proof of Proposition 2.B.4. Using the long exact sequence associated to (2.B.10) and Lemma 2.B.5 we already know

$$\begin{aligned} \mathrm{HH}^0(\mathcal{A}) &\cong \mathrm{HH}^0(X) = \mathbb{C}, \\ \mathrm{HH}^1(\mathcal{A}) &\cong \mathrm{HH}^1(X) = 0, \\ \mathrm{HH}^2(\mathcal{A}) &\cong \mathrm{HH}^2(X) = \mathbb{C}^{12}, \\ \mathrm{HH}^i(\mathcal{A}) &\cong \mathrm{NHH}^{i+1}(\mathcal{E}, X) \cong E_\infty^{i+1} \text{ for } i \geq 3. \end{aligned}$$

Moreover, as argued in Proposition 2.B.4 the spectral sequence

$$E_1^{-p,q} \Rightarrow \mathrm{NHH}^{q-p}(\mathcal{E}, X)$$

degenerates on the E_1-page. Thus, it remains to compute dimensions of the nontrivial pieces of the E_1-page. These are:

$$\begin{aligned} E_1^{-1,5} = &\left(\bigoplus_{i=1}^{10} \mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_X(D_i)) \otimes \mathrm{Ext}^{3-2}(\mathcal{O}_X(D_i), \mathcal{O}_X(-K_X)) \right) \\ &\oplus \mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_X(F)) \otimes \mathrm{Ext}^{3-2}(\mathcal{O}_X(F), \mathcal{O}_X(-K_X)) \\ &\oplus \mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_X(2F)) \otimes \mathrm{Ext}^{3-2}(\mathcal{O}_X(2F), \mathcal{O}_X(-K_X)) \\ &\oplus \left(\bigoplus_{i=1}^{10} \mathrm{Ext}^2(\mathcal{O}_X(D_i), \mathcal{O}_X(F)) \otimes \mathrm{Ext}^{3-2}(\mathcal{O}_X(F), \mathcal{O}_X(D_i - K_X)) \right) \\ &\oplus \left(\bigoplus_{i=1}^{10} \mathrm{Ext}^2(\mathcal{O}_X(D_i), \mathcal{O}_X(2F)) \otimes \mathrm{Ext}^{3-2}(\mathcal{O}_X(2F), \mathcal{O}_X(D_i - K_X)) \right) \\ &\oplus \mathrm{Ext}^2(\mathcal{O}_X(F), \mathcal{O}_X(2F)) \otimes \mathrm{Ext}^{3-2}(\mathcal{O}_X(2F), \mathcal{O}_X(F - K_X)), \end{aligned}$$

which implies that $\dim E_1^{-1,5} = 1\cdot 2\cdot 10 + 3\cdot 4 + 6\cdot 7 + 2\cdot 3\cdot 10 + 5\cdot 6\cdot 10 + 3\cdot 4 = 446$.

$$
\begin{aligned}
E_1^{-2,7} =& \left(\bigoplus_{i=1}^{10} \operatorname{Ext}^2(\mathcal{O}_X, \mathcal{O}_X(D_i)) \otimes \operatorname{Ext}^2(\mathcal{O}_X(D_i), \mathcal{O}_X(F)) \otimes \operatorname{Ext}^{3-2}(\mathcal{O}_X(F), \mathcal{O}_X(-K_X))\right) \\
&\oplus \left(\bigoplus_{i=1}^{10} \operatorname{Ext}^2(\mathcal{O}_X, \mathcal{O}_X(D_i)) \otimes \operatorname{Ext}^2(\mathcal{O}_X(D_i), \mathcal{O}_X(2F)) \right.\\
&\qquad \left. \otimes \operatorname{Ext}^{3-2}(\mathcal{O}_X(2F), \mathcal{O}_X(-K_X))\right) \\
&\oplus \operatorname{Ext}^2(\mathcal{O}_X, \mathcal{O}_X(F)) \otimes \operatorname{Ext}^2(\mathcal{O}_X(F), \mathcal{O}_X(2F)) \otimes \operatorname{Ext}^{3-2}(\mathcal{O}_X(2F), \mathcal{O}_X(-K_X)) \\
&\oplus \left(\bigoplus_{i=1}^{10} \operatorname{Ext}^2(\mathcal{O}_X(D_i), \mathcal{O}_X(F)) \otimes \operatorname{Ext}^2(\mathcal{O}_X(F), \mathcal{O}_X(2F)) \right.\\
&\qquad \left. \otimes \operatorname{Ext}^{3-2}(\mathcal{O}_X(2F), \mathcal{O}_X(D_i - K_X))\right),
\end{aligned}
$$

which implies that $\dim E_1^{-2,7} = 1\cdot 2\cdot 4\cdot 10 + 1\cdot 5\cdot 7\cdot 10 + 3\cdot 3\cdot 7 + 2\cdot 3\cdot 6\cdot 10 = 853$.

$$
\begin{aligned}
E_1^{-3,9} = \bigoplus_{i=1}^{10} &\operatorname{Ext}^2(\mathcal{O}_X, \mathcal{O}_X(D_i)) \otimes \operatorname{Ext}^2(\mathcal{O}_X(D_i), \mathcal{O}_X(F)) \\
&\otimes \operatorname{Ext}^2(\mathcal{O}_X(F), \mathcal{O}_X(2F)) \otimes \operatorname{Ext}^{3-2}(\mathcal{O}_X(2F), \mathcal{O}_X(-K_X)),
\end{aligned}
$$

which implies that $\dim E_1^{-3,9} = 1\cdot 2\cdot 3\cdot 7\cdot 10 = 420$. Finally, we conclude

$$
\begin{aligned}
&\dim \mathrm{HH}^3(\mathcal{A}) = \dim \mathrm{NHH}^4(\mathcal{E}, X) = \dim E_1^{-1,5} = 446, \\
&\dim \mathrm{HH}^4(\mathcal{A}) = \dim \mathrm{NHH}^5(\mathcal{E}, X) = \dim E_1^{-2,7} = 853, \text{ and} \\
&\dim \mathrm{HH}^5(\mathcal{A}) = \dim \mathrm{NHH}^6(\mathcal{E}, X) = \dim E_1^{-3,9} = 420.
\end{aligned}
$$

□

References

[Bei78] Aleksandr A. Beilinson. “Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra”. *Funktsional. Anal. i Prilozhen.* 12.3 (1978), pp. 68–69.

[BGKP15] Christian Böhning, Hans-Christian Graf von Bothmer, Ludmil Katzarkov, and Pawel Sosna. “Determinantal Barlow surfaces and phantom categories”. *J. Eur. Math. Soc. (JEMS)* 17.7 (2015), pp. 1569–1592.

[BO01] Alexei Bondal and Dmitri O. Orlov. “Reconstruction of a variety from the derived category and groups of autoequivalences”. *Compositio Math.* 125.3 (2001), pp. 327–344.

[BP93] Alexey I. Bondal and Alexander E. Polishchuk. “Homological properties of associative algebras: the method of helices”. *Izv. Ross. Akad. Nauk Ser. Mat.* 57.2 (1993), pp. 3–50.

[BK24] Lev Borisov and Kimoi Kemboi. “Non-existence of phantoms on some non-generic blowups of the projective plane” (2024). arXiv: 2405.01683v1.

[CHS23] Wen Chang, Fabian Haiden, and Sibylle Schroll. “Braid group actions on branched coverings and full exceptional sequences” (2023). arXiv: 2301.04398v2.

[CM11] Ciro Ciliberto and Rick Miranda. “Homogeneous interpolation on ten points”. *J. Algebraic Geom.* 20.4 (2011), pp. 685–726.

[CS99] John H. Conway and Neil J. A. Sloane. *Sphere packings, lattices and groups.* 3rd ed. Vol. 290. Grundlehren der mathematischen Wissenschaften. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. Springer-Verlag, New York, 1999. lxxiv+703.

[Dec+21] Wolfram Decker, Gert-Martin Greuel, Gerhard Pfister, and Hans Schönemann. SINGULAR *4-2-1 — A computer algebra system for polynomial computations.* 2021. URL: http://www.singular.uni-kl.de.

[Dol12] Igor V. Dolgachev. *Classical algebraic geometry.* A modern view. Cambridge University Press, Cambridge, 2012. xii+639.

[DI09] Igor V. Dolgachev and Vasily A. Iskovskikh. “Finite subgroups of the plane Cremona group”. *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I.* Vol. 269. Progr. Math. Birkhäuser Boston, Boston, MA, 2009, pp. 443–548.

[DO88] Igor V. Dolgachev and David Ortland. “Point sets in projective spaces and theta functions”. *Astérisque* 165 (1988).

[DJ07] Marcin Dumnicki and Witold Jarnicki. “New effective bounds on the dimension of a linear system in $\mathbb{P}^2$”. *J. Symbolic Comput.* 42.6 (2007), pp. 621–635.

[Efi23] Alexander Efimov. *Private communication.* Bielefeld, June 13, 2023.

[EL16] Alexey Elagin and Valery Lunts. “On full exceptional collections of line bundles on del Pezzo surfaces”. *Mosc. Math. J.* 16.4 (2016), pp. 691–709.

[EXZ21] Alexey Elagin, Junyan Xu, and Shizhuo Zhang. “On cyclic strong exceptional collections of line bundles on surfaces”. *Eur. J. Math.* 7.1 (2021), pp. 69–115.

[Fer05] Tommaso de Fernex. "Negative curves on very general blow-ups of $\mathbb{P}^2$". *Projective varieties with unexpected properties.* Walter de Gruyter, Berlin, 2005, pp. 199–207.

[Gab62] Pierre Gabriel. "Des catégories abéliennes". *Bull. Soc. Math. France* 90 (1962), pp. 323–448.

[GO13] Sergey Gorchinskiy and Dmitri O. Orlov. "Geometric phantom categories". *Publ. Math. Inst. Hautes Études Sci.* 117 (2013), pp. 329–349.

[Gor88] Alexey L. Gorodentsev. "Exceptional bundles on surfaces with a moving anticanonical class". *Izv. Akad. Nauk SSSR Ser. Mat.* 52.4 (1988), pp. 740–757, 895.

[Gor94] Alexey L. Gorodentsev. "Helix theory and nonsymmetrical bilinear forms". *Algebraic geometry and its applications (Yaroslavl', 1992).* Aspects Math., E25. Friedr. Vieweg, Braunschweig, 1994, pp. 47–59.

[GK04] Alexey L. Gorodentsev and Sergej A. Kuleshov. "Helix theory". *Mosc. Math. J.* 4.2 (2004), pp. 377–440, 535.

[Har85] Brian Harbourne. "Blowings-up of $\mathbf{P}^2$ and their blowings-down". *Duke Math. J.* 52.1 (1985), pp. 129–148.

[HP11] Lutz Hille and Markus Perling. "Exceptional sequences of invertible sheaves on rational surfaces". *Compositio Math.* 147.4 (2011), pp. 1230–1280.

[Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory.* Vol. 9. Graduate Texts in Mathematics. Second printing, revised. Springer-Verlag, New York-Berlin, 1978. xii+171.

[Huy06] Daniel Huybrechts. *Fourier-Mukai transforms in algebraic geometry.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006. viii+307.

[IOU21] Akira Ishii, Shinnosuke Okawa, and Hokuto Uehara. "Exceptional collections on Σ_2" (2021). arXiv: 2107.03051v1.

[Kap88] Mikhail M. Kapranov. "On the derived categories of coherent sheaves on some homogeneous spaces". *Invent. Math.* 92.3 (1988), pp. 479–508.

[Kol96] János Kollár. *Rational curves on algebraic varieties.* Vol. 32. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1996. viii+320.

[Kra24a] Johannes Krah. "A phantom on a rational surface". *Invent. Math.* 235.3 (2024), pp. 1009–1018.

[Kra24b] Johannes Krah. "Mutations of numerically exceptional collections on surfaces". *Math. Z.* 307.4 (2024), Paper No. 78.

[Kul97] Sergej A. Kuleshov. "Exceptional and rigid sheaves on surfaces with anti-canonical class without base components". *J. Math. Sci. (New York)* 86.5 (1997), pp. 2951–3003.

[KO94] Sergej A. Kuleshov and Dmitri O. Orlov. "Exceptional sheaves on Del Pezzo surfaces". *Izv. Ross. Akad. Nauk Ser. Mat.* 58.3 (1994), pp. 53–87.

[Kuz09] Alexander Kuznetsov. "Hochschild homology and semiorthogonal decompositions" (2009). arXiv: 0904.4330v1.

[Kuz14] Alexander Kuznetsov. "Semiorthogonal decompositions in algebraic geometry". *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II.* Kyung Moon Sa, Seoul, 2014, pp. 635–660.

[Kuz15] Alexander Kuznetsov. “Height of exceptional collections and Hochschild cohomology of quasiphantom categories”. *J. Reine Angew. Math.* 708 (2015), pp. 213–243.

[Kuz17] Alexander Kuznetsov. “Exceptional collections in surface-like categories”. *Mat. Sb.* 208.9 (2017), pp. 116–147.

[KP16] Alexander Kuznetsov and Alexander Polishchuk. “Exceptional collections on isotropic Grassmannians”. *J. Eur. Math. Soc. (JEMS)* 18.3 (2016), pp. 507–574.

[Laf] Antonio Laface. *Personal Homepage.* Last visited on June 24, 2024. URL: https://www2.udec.cl/~alaface/software.html.

[LU03] Antonio Laface and Luca Ugaglia. “Quasi-homogeneous linear systems on $\mathbb{P}^2$ with base points of multiplicity 5”. *Canad. J. Math.* 55.3 (2003), pp. 561–575.

[MT15] Matilde Marcolli and Gonçalo Tabuada. “From exceptional collections to motivic decompositions via noncommutative motives”. *J. Reine Angew. Math.* 701 (2015), pp. 153–167.

[Muk81] Shigeru Mukai. “Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves”. *Nagoya Math. J.* 81 (1981), pp. 153–175.

[Nag60] Masayoshi Nagata. “On rational surfaces. II”. *Mem. Coll. Sci. Univ. Kyoto Ser. A. Math.* 33 (1960/61), pp. 271–293.

[Orl92] Dmitri O. Orlov. “Projective bundles, monoidal transformations, and derived categories of coherent sheaves”. *Izv. Ross. Akad. Nauk Ser. Mat.* 56.4 (1992), pp. 852–862.

[Orl05] Dmitri O. Orlov. “Derived categories of coherent sheaves, and motives”. *Uspekhi Mat. Nauk* 60.6(366) (2005), pp. 231–232.

[Orl20] Dmitri O. Orlov. “Finite-dimensional differential graded algebras and their geometric realizations”. *Adv. Math.* 366 (2020), Paper No. 107096, 33.

[Per18] Markus Perling. “Combinatorial aspects of exceptional sequences on (rational) surfaces”. *Math. Z.* 288.1-2 (2018), pp. 243–286.

[Pir23] Dmitrii Pirozhkov. “Admissible subcategories of del Pezzo surfaces”. *Adv. Math.* 424 (2023), Paper No. 109046, 62.

[Swa96] Richard G. Swan. “Hochschild cohomology of quasiprojective schemes”. *J. Pure Appl. Algebra* 110.1 (1996), pp. 57–80.

[Tae23] Lenny Taelman. “Derived equivalences of hyperkähler varieties”. *Geom. Topol.* 27.7 (2023), pp. 2649–2693.

[dTVdB16] Louis de Thanhoffer de Volcsey and Michel Van den Bergh. “On an analogue of the Markov equation for exceptional collections of length 4” (2016). arXiv: 1607.04246v1.

[Via17] Charles Vial. “Exceptional collections, and the Néron-Severi lattice for surfaces”. *Adv. Math.* 305 (2017), pp. 895–934.

[Vin75] Èrnest B. Vinberg. “Some arithmetical discrete groups in Lobačevskiĭ spaces”. *Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973).* Oxford Univ. Press, Bombay, 1975, pp. 323–348.