AUTOEQUIVALENCES OF BLOW-UPS OF MINIMAL SURFACES

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Abstract. Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of points in very general position. We show that X has only standard autoequivalences, no nontrivial Fourier–Mukai partners, and admits no spherical objects. Further, we show that the same result holds if X is a blow-up of finitely many points in a minimal surface of nonnegative Kodaira dimension which contains no (-2)-curves. Independently, we characterize spherical objects on blow-ups of minimal surfaces of positive Kodaira dimension.

1. Introduction

Let X be a smooth projective variety over the complex numbers and denote by $\mathsf{D}^b(X)$ the bounded derived category of coherent sheaves on X. If the canonical bundle ω_X is ample or anti-ample, then, by Bondal–Orlov [BO01], the group of autoequivalences $\operatorname{Aut}(\mathsf{D}^b(X))$ only consists of so-called *standard autoequivalences*, i.e.

$$\operatorname{Aut}(\mathsf{D}^b(X)) = \operatorname{Pic}(X) \rtimes \operatorname{Aut}(X) \times \mathbb{Z}[1].$$

In general, the standard autoequivalences $\operatorname{Pic}(X) \rtimes \operatorname{Aut}(X) \times \mathbb{Z}[1]$ form a subgroup of $\operatorname{Aut}(\mathsf{D}^b(X))$ and $\mathsf{D}^b(X)$ often admits of non-standard autoequivalences, see, e.g., [Orl02] for the case of abelian surfaces and [BB17] for the case of $K3$ surfaces of Picard rank 1. A natural source for non-standard autoequivalences are so-called *spherical twists* [ST01].

In contrast to the case of varieties with trivial canonical class, a spherical object on a variety with nontrivial and non-torsion canonical class has to be supported on a proper closed subset, see Lemma 2.2. If X is a certain toric surface [BP14, Thm. 1, Thm. 2] or a surface of general type whose canonical model has at worst A_n-singularities [IU05, Thm. 1.5], then $\operatorname{Aut}(\mathsf{D}^b(X))$ is generated by standard equivalences and spherical twists.

In Sections 3 and 4, we focus on rational surfaces X which are blow-ups of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of points in very general position. It follows from [Fer05, Prop. 2.2], recalled in Proposition 2.3, that such a surface X does not contain any (-2)-curve. Motivated by the results of [IU05], it is reasonable to expect that the absence of (-2)-curves implies the absence of spherical objects. We confirm this expectation by arguing that a spherical object on X has to be supported on a union of *rational* integral curves, see Lemma 3.2. Moreover, we obtain the following

Theorem 1.1. *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of points in very general position. Then the following statements hold:*

(i) *Any autoequivalence of X is standard, i.e.* $\operatorname{Aut}(\mathsf{D}^b(X)) = \operatorname{Pic}(X) \rtimes \operatorname{Aut}(X) \times \mathbb{Z}[1]$.
(ii) *If Y is a smooth projective variety such that $\mathsf{D}^b(X) \cong \mathsf{D}^b(Y)$, then $X \cong Y$.*
(iii) *There exists no spherical object in $\mathsf{D}^b(X)$.*

By [Fav12, Cor. 4.4], for any smooth projective variety X we have that (i) implies (ii). Moreover, if X is a smooth projective variety of dimension ≥ 2 with nontrivial and non-torsion canonical class,

Date: October 27, 2023.
2020 *Mathematics Subject Classification.* 14F08, 14J26.
Key words and phrases. Derived Categories, Autoequivalences, Rational Surfaces, Spherical Objects.
The research of both authors was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – SFB-TRR 358/1 2023 – 491392403. Further, the research of X.H. was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 516701553, and the research of J.K. was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC-2047/1 – 390685813.

then (i) implies (iii). Indeed, arguing as in Lemma 2.2, a spherical object S on such a variety X has to be supported on a proper closed subvariety. By [Huy06, Ex. 8.5 (ii)], the spherical twist T_S associated to S satisfies $T_S(S) = S[1-\dim X]$ and $T_S(k(x)) = k(x)$ for any point $x \in X \setminus \operatorname{Supp}(S)$. Thus, T_S is a non-standard autoequivalence.

We provide two proofs of Theorem 1.1, both utilizing the results of de Fernex [Fer05] regarding rational curves in the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of points in very general position. The first proof relies on the geometric observation that a curve $C \subseteq X$ such that $K_X|_C$ is trivial in $\mathrm{CH}_0(C)_{\mathbb{Q}}$ is rational; see Lemma 3.2. This allows to give a direct proof of each statement in Theorem 1.1 (although, as explained above, it would suffice to prove (i) by using the result of Favero). The second proof, outlined in Section 4, relies on Uehara's more general classification results and his description of autoequivalence groups of surfaces with Fourier–Mukai support dimension 2 satisfying a condition on the configuration of (-2)-curves [Ueh19].

In Section 5 we consider blow-ups X of minimal surfaces Y of nonnegative Kodaira dimension. In contrast to the case of rational surfaces, (-2)-curves on X are strict transforms of (-2)-curves on Y, see Proposition 5.1. Thus, using [Ueh19], we obtain

Theorem 1.2 (Theorem 5.2)**.** *Let Y be a minimal surface of nonnegative Kodaira dimension and let X be the blow-up of Y in a nonempty finite set of points. Assume Y contains no (-2)-curves, e.g. Y has Kodaira dimension 1 and the elliptic fibration of Y has only irreducible fibers. Then $\mathsf{D}^b(X)$ admits only standard autoequivalences, i.e.*

$$\operatorname{Aut}(\mathsf{D}^b(X)) = \operatorname{Pic}(X) \rtimes \operatorname{Aut}(X) \times \mathbb{Z}[1].$$

As outlined above, Theorem 1.2 implies that such X has no Fourier–Mukai partners and $\mathsf{D}^b(X)$ does not contain spherical objects.

In Proposition 5.7 we characterize spherical objects on blow-ups X of minimal surfaces Y of positive Kodaira dimension: An object in $\mathsf{D}^b(X)$ is spherical if and only if it is the pullback of a spherical object in $\mathsf{D}^b(Y)$ whose support is disjoint from the exceptional locus of $X \to Y$. If Y is a minimal surface of Kodaira dimension 1 whose elliptic fibration has only irreducible fibers, this characterization combined with the results of [Ueh16] gives an alternate proof that $\mathsf{D}^b(X)$ does not contain spherical objects, see Remark 5.9.

Acknowledgements. The authors thank Hokuto Uehara and Charles Vial for useful comments on an earlier draft of this paper. Further, the authors thank Gebhard Martin for helpful discussions regarding elliptic surfaces.

Conventions. The term *surface* always refers to a smooth projective 2-dimensional variety over $\mathbb{C}$. For a variety X, we denote by $\mathrm{CH}^*(X)$ (resp. $\mathrm{CH}_*(X)$) the Chow groups of algebraic cycles modulo rational equivalence with integer coefficients graded by codimension (resp. dimension). We denote by $\mathrm{CH}^*(X)_{\mathbb{Q}} := \mathrm{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ the Chow groups with rational coefficients. A $(-k)$-curve C in a surface S is an integral smooth rational curve C with self-intersection number $-k$.

2. Preliminary Observations

Let X be a smooth projective variety. The support of an object $F \in \mathsf{D}^b(X)$ is by definition the closed subvariety

$$\operatorname{Supp}(F) := \bigcup_{i \in \mathbb{Z}} \operatorname{Supp}(\mathcal{H}^i(F)) \subseteq X$$

endowed with the unique reduced closed subscheme structure. If F is a simple object, i.e. $\operatorname{Hom}(F, F) = \mathbb{C}$, then $\operatorname{Supp}(F)$ is connected; see, e.g., [Huy06, Lem. 3.9].

Definition 2.1. An object $S \in \mathsf{D}^b(X)$ is called *spherical* if

$$\operatorname{Hom}(S, S[i]) = \begin{cases} \mathbb{C} & \text{if } i = 0, \dim X, \\ 0 & \text{else,} \end{cases}$$

and $S \otimes \omega_X \cong S$.

Denote by $p, q \colon X \times X \to X$ the projections and by $\Delta \hookrightarrow X \times X$ the diagonal embedding. If S is a spherical object on X, the object $\mathcal{P}_S := \operatorname{Cone}(\mathrm{L}q^*S^\vee \otimes^{\mathrm{L}} \mathrm{L}p^*S \to \mathcal{O}_\Delta) \in \mathsf{D}^b(X \times X)$ is the Fourier–Mukai kernel of the *spherical twist* $T_S \colon \mathsf{D}^b(X) \to \mathsf{D}^b(X)$ given by $T_S(-) = \mathrm{R}p_*(\mathcal{P}_S \otimes^{\mathrm{L}} \mathrm{L}q^*(-))$.

Note that by [ST01, Thm. 1.2] a spherical twist is always an autoequivalence of $\mathsf{D}^b(X)$. The condition $S \otimes \omega_X \cong S$ has the following consequence on the support of a spherical object:

Lemma 2.2. *Let X be a smooth projective positive dimensional variety with $K_X \neq 0$ in $\mathrm{CH}^*(X)_\mathbb{Q}$, i.e. ω_X is nontrivial and non-torsion. Then any spherical object $S \in \mathsf{D}^b(X)$ is supported on a connected proper closed subset.*
Moreover, if X is a surface, then $\operatorname{Supp}(S)$ is a, possibly reducible, connected curve $C = \bigcup_i C_i$ such that $K_X|_{\tilde{C}_i} = 0$ in $\mathrm{CH}^1(\tilde{C}_i)_\mathbb{Q}$, where C_i are the irreducible components of C and $\tilde{C}_i \to C_i$ are the normalizations. In particular, $K_X \cdot C = 0 \in \mathbb{Z}$.

Proof. Denote by $\mathcal{H}^i(S) \in \operatorname{Coh} X$ the i-th cohomology sheaf of S. Since ω_X is a line bundle, we have
$$\mathcal{H}^i(S) \otimes \omega_X = \mathcal{H}^i(S \otimes \omega_X) \cong \mathcal{H}^i(S),$$
which yields $\operatorname{ch}(\mathcal{H}^i(S))\operatorname{ch}(\omega_X) = \operatorname{ch}(\mathcal{H}^i(S))$ in $\mathrm{CH}^*(X)_\mathbb{Q}$. If $\mathcal{H}^i(S)$ had positive rank, then $\operatorname{ch}(\mathcal{H}^i(S))$ would be invertible in $\mathrm{CH}^*(X)_\mathbb{Q}$, hence $\operatorname{ch}(\omega_X) = 0$. This contradicts to K_X being non-torsion. Hence, all cohomology sheaves $\mathcal{H}^i(S)$ have rank zero and thus the generic point of X is not contained in the support of S. Thus, $\dim \operatorname{Supp}(S) < \dim X$ and $\operatorname{Supp}(S)$ is connected by [Huy06, Lem. 3.9].

Assume in addition that $\dim S = 2$. If S were supported on a point, then [Huy06, Lem. 4.5] would show that $S \cong k(x)[m]$ for some $x \in X$ and $m \in \mathbb{Z}$. In particular, $\chi(k(x)[m], k(x)[m]) = 0$, but $\chi(S, S) = 2$. Hence, $\operatorname{Supp}(S)$ is 1-dimensional and connected, i.e. a connected reduced, possibly reducible, curve.

Let $C_i \subseteq X$ be an irreducible curve, contained in $\operatorname{Supp}(S)$ and let $\tilde{C}_i \to C_i$ be its normalization. Denoting by $j \colon \tilde{C}_i \to C_i \hookrightarrow X$ composition, we obtain by the projection formula
$$K_X \cdot C_i = j_* j^* K_X \in \mathrm{CH}_0(X).$$
Let $\mathcal{H}$ be a cohomology sheaf of S which has nonzero rank restricted on C_i. The equality $\operatorname{ch}(\mathcal{H}) = \operatorname{ch}(\mathcal{H})\operatorname{ch}(\omega_X)$ on X shows $\operatorname{ch}(j^*\mathcal{H}) = \operatorname{ch}(j^*\mathcal{H})\operatorname{ch}(j^*\omega_X)$ on $\tilde{C}_i$. Since $j^*\mathcal{H}$ has nonzero rank, this implies that j^*K_X is torsion in $\mathrm{CH}_0(\tilde{C}_i)$. We conclude that the intersection number $K_X \cdot C_i = \deg(j_* j^* K_X)$ is zero. □

Let X be the blow-up of $\mathbb{P}^2_\mathbb{C}$ in a finite set of points in very general position. The following result of de Fernex shows that X contains no integral rational curves of self-intersection less or equal than -2.

Proposition 2.3 ([Fer05, Prop. 2.3])**.** *Let X be the blow-up of $\mathbb{P}^2_\mathbb{C}$ in a finite set of points in very general position. If $C \subseteq X$ is an integral rational curve with $C^2 < 0$, then C is a (-1)-curve, that is a smooth rational curve of self-intersection -1.*

Moreover, the following Proposition 2.4 follows from the proof of [Fer05, Prop. 2.4].

Proposition 2.4 ([Fer05])**.** *Let X be the blow-up of $\mathbb{P}^2_\mathbb{C}$ in a finite set of points in very general position. If $C \subseteq X$ is an integral rational curve, then $C \cdot K_X < 0$.*

Therefore X cannot contain an integral rational curve $C \subseteq X$ such that $C \cdot K_X = 0$.

3. Proof of Theorem 1.1

In Lemma 2.2 we have seen that a spherical object in $\mathsf{D}^b(X)$ is supported on a curve $C \subseteq X$ such that $C_i \cdot K_X = 0$ for every irreducible component C_i of C. The proof of Theorem 1.1 relies on a refinement of this observation, namely that every such curve C_i is rational.

Recall the following construction from [Voi03, § 10]: Let X be a projective variety over $\mathbb{C}$ and denote by $X^{(d)}$ the d-th symmetric product of X. Let $c \colon X^{(d)} \to \mathrm{CH}_0(X)$ be the map defined by
$$X^{(d)} \ni Z \mapsto \text{class of } Z \text{ mod rational equivalence}.$$

Further define

$$\sigma_d \colon X^{(d)} \times X^{(d)} \to \mathrm{CH}_0(X)_{\mathrm{hom}}$$
$$(Z_1, Z_2) \mapsto c(Z_1) - c(Z_2),$$

where $\mathrm{CH}_0(X)_{\mathrm{hom}} \subseteq \mathrm{CH}_0(X)$ denotes the subspace of homologically trivial cycles.

Lemma 3.1 ([Voi03, Lem. 10.7]). *The fibers of σ_d are countable unions of closed algebraic subsets of $X^{(d)} \times X^{(d)}$.*

Lemma 3.2. *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in n points in general position. If $C \subseteq X$ is an integral curve with $K_X|_{\tilde{C}} = 0 \in \mathrm{CH}^1(\tilde{C})_{\mathbb{Q}}$, where $\tilde{C} \to C$ is the normalization, then C is rational.*

Proof. We denote by E_i the exceptional divisor over the i-th blown up point. Then $K_X = -3H + \sum_i E_i$ and by assumption $mK_X|_C = 0 \in \mathrm{CH}^1(C)_{\mathbb{Q}}$ for all $m \in \mathbb{Z}$. Note that C cannot be one of the exceptional curves E_i, since $K_X \cdot E_i = -1$ for all i. Hence, C is the strict transform of a curve of degree $d = H \cdot C$. Since $mK_X \cdot C = 0$, the intersection of $m \sum_i E_i$ and C defines a unique point in the symmetric product $Z_2 := (x_1, \ldots, x_{3md}) \in C^{(3md)}$.

Consider the set

$$|3mH| \cap C := \{C' \cap C \mid C' \in |3mH| \text{ such that } C \not\subseteq C'\}$$

as a subset of $C^{(3md)}$. We claim that for sufficiently large $m > 0$ the subset $|3mH| \cap C$ is dense in $C^{(3md)}$.

Indeed, let $q_1, \ldots, q_{3md} \in C \setminus E_1 \cup \cdots \cup E_n$ be pairwise distinct points and let $X' \to X$ be the blow-up of $q_1, \ldots, q_{3md}$. Denote by E'_i the exceptional divisor over the point q_i for $1 \leq i \leq 3md$ and consider the divisor

$$D := 3mH - \sum_{i=1}^{3md} E'_i \quad \text{on } X'.$$

A member of the linear system $|D|$ can be identified with a curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree $3m$ vanishing at the points $q_1, \ldots, q_{3md}$. By Riemann–Roch

$$\chi(D) = 1 + \frac{1}{2} D \cdot (D - K_{X'}) = 1 + \frac{1}{2}(9m^2 + 9m - 6md),$$

thus $\chi(D) > 0$ for sufficiently large $m > 0$. Since Serre duality shows

$$h^2(X', \mathcal{O}_{X'}(D)) = h^0(X', \mathcal{O}_{X'}(-D + K_{X'})) = h^0\left(X', \mathcal{O}_{X'}\left(-3(m+1)H + \sum_{i=1}^{n} E_i\right)\right) = 0,$$

we have $h^0(X', \mathcal{O}_{X'}(D)) \geq \chi(D) > 0$ for sufficiently large $m > 0$. It follows from [Mig01, Thm. 1] that a general member C' of $|D|$ is smooth and irreducible for sufficiently large $m > 0$. Hence, $C \not\subseteq C'$ and $C \cap C' = \{q_1, \ldots, q_{3md}\}$. This shows that $|3mH| \cap C$ contains a Zariski open subset of $C^{(3md)}$ for sufficiently large $m > 0$. For the rest of the proof we fix such $m > 0$.

We first assume that C is smooth. Further, we assume for contradiction that C is not rational. Let $\bar{\sigma}_{3md}$ be the restriction of $\sigma_{3md} \colon C^{(3md)} \times C^{(3md)} \to \mathrm{CH}_0(C)_{\mathrm{hom}}$ to $C^{(3md)} \times \{Z_2\}$. By Lemma 3.1, for every $t \in \mathrm{CH}_0(C)$ the fiber $\bar{\sigma}^{-1}_{3md}(t)$ is a countable union of closed algebraic subsets. We denote by $\mathrm{CH}_0(C)_{\mathrm{tor}}$ the torsion classes in $\mathrm{CH}_0(C)$. Recall that $\mathrm{CH}_0(C)_{\mathrm{tor}}$ is countable, thus $\bigcup_{t \in \mathrm{CH}_0(C)_{\mathrm{tor}}} \bar{\sigma}^{-1}_{3md}(t)$ is also a countable union of closed algebraic subsets. Let

$$Z_1 := Z_2 - x_{3md} + y = x_1 + \cdots + x_{3md-1} + y,$$

where $y \in C$ is a point such that $c(y) - c(x_{3md})$ is not torsion in $\mathrm{CH}_0(C)$. Note that such y exists since there are only countable many torsion points in $\mathrm{CH}_0(C)$ and for every $x \neq y \in C$, $c(x) \neq c(y) \in \mathrm{CH}_0(C)$. Hence, $\bigcup_{t \in \mathrm{CH}_0(C)_{\mathrm{tor}}} \bar{\sigma}^{-1}_{3md}(t) \subseteq C^{(3md)}$ is a countable union of proper closed algebraic subsets. We have argued above that $|3mH| \cap C$ contains a Zariski open subset of $C^{(3md)}$, thus a very general member $Z \in |3mH| \cap C$ satisfies $\sigma_{3md}(Z, Z_2) \neq 0$ in $\mathrm{CH}_0(C)_{\mathbb{Q}}$. Hence, $mK_X|_C \neq 0$ in $\mathrm{CH}_0(C)_{\mathbb{Q}}$. But by assumption $mK_X|_C = 0$ in $\mathrm{CH}_0(C)_{\mathbb{Q}}$, thus C has to be a rational curve. In case C is not smooth, we can argue in the same way by replacing C by its normalization and the restriction to C by the composition of restriction and pullback to the normalization. $\square$

Proof of Theorem 1.1 (iii). Assume for contradiction that $S \in \mathsf{D}^b(X)$ is a spherical object. By Lemma 2.2, S is supported on a connected curve $C = C_i$ such that $K_X|_{C_i} = 0 \in \mathrm{CH}^1(C_i)_{\mathbb{Q}}$ for all irreducible components C_i of C. By Lemma 3.2, each C_i is rational, thus by Proposition 2.4 $C_i \cdot K_X \neq 0$. This contradicts to $K_X|_{C_i} = 0$ in $\mathrm{CH}^1(C_i)_{\mathbb{Q}}$. □

Proof of Theorem 1.1 (i) and (ii). Let $\phi\colon \mathsf{D}^b(Y) \to \mathsf{D}^b(X)$ be an equivalence. For any point $y \in Y$ the skyscraper sheaf $k(y)$ satisfies $k(y) \otimes \omega_Y \cong k(y)$ and thus $\phi(k(y)) \otimes \omega_X \cong \phi(k(y))$. Moreover, since $\mathbb{C} = \mathrm{Hom}(k(y), k(y)) = \mathrm{Hom}(\phi(k(y)), \phi(k(y)))$, [Huy06, Lem. 3.9] shows that $\mathrm{Supp}(\phi(k(y))$ is connected. Arguing as in Lemma 2.2, we observe that $\mathrm{Supp}(\phi(k(y)))$ is either a point or $\mathrm{Supp}(\phi(k(y))) = \bigcup_i C_i$, where each C_i is an integral curve with $K_X|_{C_i} = 0 \in \mathrm{CH}^1(C_i)_{\mathbb{Q}}$. In the latter case each C_i is rational by Lemma 3.2. By Proposition 2.4, $C_i \cdot K_X \neq 0$. Hence, $\phi(k(y))$ is supported on a point $x \in X$ and by [Huy06, Lem. 4.5] $\phi(k(y)) = k(x)[m]$ for some $m \in \mathbb{Z}$. Moreover, by [Huy06, Cor. 6.14] the locus of $y \in Y$ such that $\phi \circ [-m](k(y))$ is a skyscraper sheaf is open. Since Y is connected, this locus is the whole of Y, which shows that the shift m in $\phi(k(y)) = k(x)[m]$ is independent of $y \in Y$. Thus $\phi \circ [-m]$ sends skyscraper sheaves to skyscraper sheaves and [BM17, § 3.3] (or [Huy06, Cor. 5.23]) shows that $\phi \circ [-m] = f_*(\mathcal{L} \otimes -)$ for some line bundle $\mathcal{L} \in \mathrm{Pic}(Y)$ and isomorphism $f\colon Y \to X$. This proves (ii) and shows that in the case $Y = X$ the autoequivalence ϕ is a standard autoequivalence. Thus, (i) follows. □

Remark 3.3 (On the position of blown up points). We assumed the blown up points in Theorem 1.1 to be in very general position. On the one hand, this is required in de Fernex' Proposition 2.3 to ensure that X admits no (-2)-curves. On the other hand, Lemma 3.2 relies on [Mig01, Thm. 1] which requires the blown up points to be in general position.

4. Alternative Proof of Theorem 1.1 (i) and (ii)

An alternative proof of Theorem 1.1 (i) and (ii), which is more dependent on the literature, can be obtained using [Ueh19, Thm. 1, Thm. 2] and [Kaw02] as we outline in the following:

Second proof of Theorem 1.1 (i) and (ii). Recall, e.g. from [CD12, Prop. 2.2], that if Y is a rational surface admitting a minimal elliptic fibration, then Y can be obtained from $\mathbb{P}^2_{\mathbb{C}}$ by blowing up 9, possibly infinitely near, points and, for some $m > 0$, the linear system $|-mK_Y|$ is a pencil. Hence, if X is the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of points in very general position, then X admits no minimal elliptic fibration. Indeed, this is clear if the number of blown up points is different from 9. In the case of 9 blown up points the linear system $|-mK_X|$ is zero-dimensional for any $m > 0$, so it is not a pencil. By [Kaw02, Thm. 1.6], a non-minimal surface admits nontrivial Fourier–Mukai partners only if it admits a minimal elliptic fibration. Hence, Theorem 1.1 (ii) follows.

Let Y be any surface and let $\Phi_P\colon \mathsf{D}^b(Y) \to \mathsf{D}^b(Y)$ be an autoequivalence with Fourier–Mukai kernel $P \in \mathsf{D}^b(Y \times Y)$. We denote by $\mathrm{Comp}(\Phi_P)$ the set of irreducible components in $\mathrm{Supp}(P) \hookrightarrow Y \times Y$ and by

$$N_Y := \max\{\dim W \mid W \in \mathrm{Comp}(\Phi_P), \Phi_P \in \mathrm{Aut}(\mathsf{D}^b(Y))\}$$

the *Fourier–Mukai support dimension* of Y. By Uehara's classification [Ueh19, Thm. 1], the equality $N_Y = 2$ is equivalent to Y admitting no minimal elliptic fibration and K_Y being not numerically equivalent to zero. Hence, for X the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of points in very general position we have $N_X = 2$.

If Y is a surface with $N_Y = 2$ such that the union of all (-2)-curves in Y forms a disjoint union of configurations of type A, then, by [Ueh19, Thm. 2], $\mathrm{Aut}(\mathsf{D}^b(Y))$ is generated by standard autoequivalences and spherical twists. For X the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of points in very general position, de Fernex' Proposition 2.3 shows that X contains no (-2)-curve. Hence, Theorem 1.1 (i) follows. □

5. Surfaces of Nonnegative Kodaira Dimension

5.1. **Autoequivalences.** In contrast to the case of negative Kodaira dimension, blowing up points in arbitrary position on minimal surfaces of nonnegative Kodaira dimension does not give rise to new (-2)-curves.

Proposition 5.1. *Let Y be a minimal surface of nonnegative Kodaira dimension and let $p\colon X \to Y$ be the blow-up of Y in a set of points $p_1, \dots, p_n \in Y$. Then every (-2)-curve C in X is the strict transform of a (-2)-curve C_0 in Y such that $p_i \notin C_0$ for $1 \leq i \leq n$.*

Proof. We denote by E_i the exceptional divisor over the i-th blown up point p_i for $1 \leq i \leq n$. Let $C \subseteq X$ be a (-2)-curve. By adjunction, we have

$$0 = g(C) = 1 + \frac{1}{2}(C^2 + C \cdot K_X),$$

where $g(C)$ denotes the geometric genus of C. Thus, $C \cdot K_X = 0$. Further, since C is not one of the exceptional curves E_i, C is the strict transform of a curve $C_0 \subseteq Y$. We have

$$0 = C \cdot K_X = C_0 \cdot K_Y + \sum_{i=1}^{n} m_i,$$

where m_i is the multiplicity of C_0 at p_i. Since K_Y is nef, each of the m_i is zero, in other words $p_i \notin C_0$ for $1 \leq i \leq n$. We conclude that C_0 is a smooth rational curve with $K_Y \cdot C_0 = 0$, hence, by adjunction, a (-2)-curve. $\square$

As a consequence of Proposition 5.1 and [Ueh19], we obtain the following

Theorem 5.2. *Let Y be a minimal surface of nonnegative Kodaira dimension and let X be the blow-up of Y in a nonempty finite set of points. Assume Y contains no (-2)-curves, e.g. Y has Kodaira dimension 1 and the elliptic fibration of Y has only irreducible fibers. Then $\mathsf{D}^b(X)$ admits only standard autoequivalences, i.e.*

$$\mathrm{Aut}(\mathsf{D}^b(X)) = \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X) \times \mathbb{Z}[1].$$

Proof. By Proposition 5.1, X contains no (-2)-curves. Thus, the statement follows from [Ueh19, Thm. 1.1, Thm. 1.3] if X admits no minimal elliptic fibration. The latter can be shown as follows: Recall, e.g. from [CDL23, Cor. 4.1.7], that a surface S with minimal elliptic fibration satisfies $K_S^2 = 0$. If $\kappa(Y) = 0$, then K_Y is numerically equivalent to zero. Hence, $K_Y^2 = 0$ and therefore $K_X^2 < 0$. If $\kappa(Y) = 1$, then Y has an elliptic fibration and therefore $K_Y^2 = 0$. Hence, $K_X^2 < 0$. Finally, if $\kappa(Y) = 2$, then X has no elliptic fibration by [Bar+04, Prop. 12.5]. $\square$

Remark 5.3. Note that the description of autoequivalences as in Theorem 5.2 is not true for a minimal surface Y. For example, if $\kappa(Y) = 1$, then $\mathrm{Aut}(\mathsf{D}^b(Y))$ can be characterized as in [Ueh16, Thm. 4.1]. In that case, as outlined in the proof of [Ueh19, Thm. 1.1], Y admits an autoequivalence $\Phi_{\mathcal{U}}$ where $\mathcal{U}$ is the universal sheaf on $Y \times J_Y(1,1)$ and $J_Y(1,1) \cong Y$ is a moduli space of stable sheaves on a smooth fiber of the elliptic fibration of Y. In this case, the support of $\mathcal{U}$ is 3-dimensional, thus $\Phi_{\mathcal{U}}$ does not lift to an autoequivalence of a blow-up of Y.

Remark 5.4 (Infinitely near points). Let X be a non-minimal surface of nonnegative Kodaira dimension with minimal model Y. If the (-2)-curves in Y only form chains of type A, then it is possible to describe $\mathrm{Aut}(\mathsf{D}^b(X))$ as in [Ueh19, Thm. 1.3]. Indeed, arguing as in [IU05, Thm. 1.5] one shows that the (-2)-curves in X only form chains of type A. Thus, [Ueh19, Thm. 1.3] applies and shows that $\mathrm{Aut}(\mathsf{D}^b(X))$ is generated by standard autoequivalences and spherical twists.

5.2. **Spherical Objects.** Similar to Proposition 5.1, spherical objects in the blow-up of a minimal surface of positive Kodaira dimension are completely determined by the minimal surface.

We begin with recalling two elementary Lemmata 5.5 and 5.6 regarding morphisms and the support of complexes of sheaves. As we were unable to find a suitable statement in the literature, we include a proof of Lemma 5.5.

Lemma 5.5. *Let X be a smooth projective variety and let $F, G \in \mathsf{D}^b(X)$.*

(i) *If $\mathrm{Supp}(F) \cap \mathrm{Supp}(G) = \emptyset$, then $\mathrm{Hom}_{\mathsf{D}^b(X)}(F, G) = 0$.*
(ii) *If $D \subseteq X$ is a divisor and $\mathrm{Supp}(F) \cap D = \emptyset$, then $F \otimes \mathcal{O}_X(D) = F$.*

Proof. We first prove (i). The condition $\operatorname{Supp}(F)\cap\operatorname{Supp}(G)=\emptyset$ implies $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{H}^{-q}(F),\mathcal{H}^l(G))=0$ for all $p,q,l\in\mathbb{Z}$. Recall, e.g. from [Huy06, p. 77], that we have a spectral sequence

$$E_2^{p,q}=\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{H}^{-q}(F),\mathcal{H}^l(G))\Rightarrow\mathcal{E}xt^{p+q}_{\mathcal{O}_X}(F,\mathcal{H}^l(G))$$

for every $l\in\mathbb{Z}$. Similarly, we have a spectral sequence

$$E_2^{p,q}=\mathcal{E}xt^p_{\mathcal{O}_X}(F,\mathcal{H}^q(G))\Rightarrow\mathcal{E}xt^{p+q}_{\mathcal{O}_X}(F,G).$$

Thus, $\operatorname{Supp}(F)\cap\operatorname{Supp}(G)=\emptyset$ implies $\mathcal{E}xt^l_{\mathcal{O}_X}(F,G)=0$ for all $l\in\mathbb{Z}$. Finally, the local-to-global spectral sequence

$$E_2^{p,q}=H^p(X,\mathcal{E}xt^q_{\mathcal{O}_X}(F,G))\Rightarrow\operatorname{Ext}^{p+q}_{\mathcal{O}_X}(F,G)$$

shows $\operatorname{Ext}^{p+q}_{\mathcal{O}_X}(F,G)=0$.

To prove (ii), assume that $D\subseteq X$ is a divisor and that $\operatorname{Supp}(F)\cap D=\emptyset$. The ideal sheaf sequence

$$0\to\mathcal{O}_X(-D)\to\mathcal{O}_X\to\mathcal{O}_D\to 0$$

yields an exact sequence

$$0\to\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_D,F)\to F\to F\otimes\mathcal{O}_X(D)\to\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_D,F)\to 0.$$

As argued above, we have $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_D,F)=0=\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_D,F)$. Hence, $F\to F\otimes\mathcal{O}_X(D)$ is an isomorphism. □

Lemma 5.6 ([BM02, Lem. 5.3])**.** *Let X be a smooth projective variety and $F\in\mathsf{D}^b(X)$. Then a point $x\in X$ lies in $\operatorname{Supp}(F)$ if and only if $\operatorname{Hom}_{\mathsf{D}^b(X)}(F,k(x)[l])\neq 0$ for some $l\in\mathbb{Z}$.*

The following Proposition 5.7 characterizes spherical objects in blow-ups of minimal surfaces of positive Kodaira dimension.

Proposition 5.7. *Let Y be a minimal surface of positive Kodaira dimension and let $p\colon X\to Y$ be the blow-up of Y in a set of points $p_1,\dots,p_n\in Y$. Then every spherical object in $\mathsf{D}^b(X)$ is of the form $\mathrm{L}p^*S$ for some spherical object $S\in\mathsf{D}^b(Y)$. Moreover, if $S\in\mathsf{D}^b(Y)$ is spherical, then $\mathrm{L}p^*S$ is spherical if and only if $p_i\notin\operatorname{Supp}(S)$ for all $1\le i\le n$.*

Proof. We denote by E_i the exceptional divisor over the i-th blown up point p_i for $1\le i\le n$. We first prove the following

Claim. If $S'\in\mathsf{D}^b(X)$ is a spherical object, then $\operatorname{Supp}(S')$ is disjoint from each E_i.

Proof of the claim. Assume $S'\in\mathsf{D}^b(X)$ is spherical, then, by Lemma 2.2, $\operatorname{Supp}(S')=\bigcup_i C_i$, where each C_i is an integral curve with $K_X\cdot C_i=0$. Since $K_X=p^*K_Y+\sum_i E_i$, such curve C_i is the strict transform of a curve in Y. Moreover, if C_0 is a curve in Y, the strict transform of C_0 has class $p^*C_0-\sum_i m_iE_i$, where m_i is the multiplicity of C_0 at p_i. We compute that

$$K_X\cdot\left(p^*C_0-\sum_{i=1}^n m_iE_i\right)=K_Y\cdot C_0+\sum_{i=1}^n m_i.$$

Since K_Y is nef, we have $K_Y\cdot C_0\ge 0$ and therefore $p_i\notin C_0$ for all $1\le i\le n$. □

Recall that $\mathsf{D}^b(X)$ admits a semiorthogonal decomposition

$$\mathsf{D}^b(X)=\langle\mathcal{O}_{E_1}(-1),\dots,\mathcal{O}_{E_n}(-1),\mathrm{L}p^*\mathsf{D}^b(Y)\rangle.$$

Since $\operatorname{Supp}(S')$ is disjoint from each E_i, we have, by Lemma 5.5,

$$\operatorname{Hom}_{\mathsf{D}^b(X)}(S',\mathcal{O}_{E_i}(-1)[l])=0=\operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{O}_{E_i}(-1),S'[l])$$

for every $l\in\mathbb{Z}$. Hence, $S'\in\mathrm{L}p^*\mathsf{D}^b(Y)$, i.e., there exists a object $S\in\mathsf{D}^b(Y)$ such that $\mathrm{L}p^*S\cong S'$. Note that $\mathrm{R}p_*\mathcal{O}_X=\mathcal{O}_Y$ implies

$$\begin{aligned}(5.8)\qquad \operatorname{Hom}_{\mathsf{D}^b(X)}(S',S'[l])&=\operatorname{Hom}_{\mathsf{D}^b(X)}(\mathrm{L}p^*S,\mathrm{L}p^*S[l])=\operatorname{Hom}_{\mathsf{D}^b(Y)}(S,\mathrm{R}p_*\mathrm{L}p^*S[l])\\&=\operatorname{Hom}_{\mathsf{D}^b(Y)}(S,S\otimes^{\mathrm{L}}\mathrm{R}p_*\mathcal{O}_X[l])=\operatorname{Hom}_{\mathsf{D}^b(Y)}(S,S[l]).\end{aligned}$$

for every $l \in \mathbb{Z}$. Moreover, since $\mathrm{Supp}(S')$ is disjoint from the exceptional divisors E_i, Lemma 5.5 shows that $\mathrm{L}p^*S \otimes \mathcal{O}_X(\sum_i E_i) = \mathrm{L}p^*S$. Hence, $\mathrm{L}p^*S \otimes p^*\omega_Y \cong \mathrm{L}p^*S$. Pushing forward via $\mathrm{R}p_*$ and using the projection formula shows that $S \otimes \omega_Y \cong S$. Thus, S is a spherical object in $\mathsf{D}^b(Y)$.

Now let $S \in \mathsf{D}^b(Y)$ be a spherical object. As in (5.8), we have

$$\mathrm{Hom}_{\mathsf{D}^b(X)}(\mathrm{L}p^*S, \mathrm{L}p^*S[l]) = \mathrm{Hom}_{\mathsf{D}^b(Y)}(S, S[l])$$

for every $l \in \mathbb{Z}$. Thus, $\mathrm{L}p^*S$ is spherical if $\mathrm{L}p^*S \otimes \omega_X \cong \mathrm{L}p^*S$. Let $x \in X$ be a point, then $\mathrm{R}p_*k(x) = k(p(x))$ and by adjunction

$$\mathrm{Hom}_{\mathsf{D}^b(X)}(S, \mathrm{R}p_*k(x)[l]) = \mathrm{Hom}_{\mathsf{D}^b(X)}(\mathrm{L}p^*S, k(x)[l])$$

for every $l \in \mathbb{Z}$. Hence, Lemma 5.6 shows that $\mathrm{Supp}(\mathrm{L}p^*S) = p^{-1}(\mathrm{Supp}(S))$. By the previous claim, it is necessary that $p^{-1}(\mathrm{Supp}(S))$ is disjoint from each E_i for $\mathrm{L}p^*S$ to be spherical. On the other hand, this is also sufficient, since $\mathrm{L}p^*S \otimes \mathcal{O}_X(\sum_i E_i) = \mathrm{L}p^*S$ holds by Lemma 5.5 if $p^{-1}(\mathrm{Supp}(S))$ is disjoint from each E_i. □

Remark 5.9. Let Y be a minimal surface of Kodaira dimension 1 whose elliptic fibration has only irreducible fibers. It follows from the description of $\mathrm{Aut}(\mathsf{D}^b(Y))$ in [Ueh16] that $\mathsf{D}^b(Y)$ does not contain spherical objects. Thus, if X is a blow-up of Y in a finite set of points, then, by Proposition 5.7, $\mathsf{D}^b(X)$ does not contain spherical objects either. Alternately, this can also be deduced from Theorem 5.2.

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