AUTOEQUIVALENCES OF BLOW-UPS OF MINIMAL SURFACES

XIANYU HU AND JOHANNES KRAH

Abstract. Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of very general points. We deduce from the work of Uehara [Ueh19] that X has only standard autoequivalences, no nontrivial Fourier–Mukai partners, and admits no spherical objects. If X is the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 9 very general points, we provide an alternate and direct proof of the corresponding statement. Further, we show that the same result holds if X is a blow-up of finitely many points in a minimal surface of nonnegative Kodaira dimension which contains no (-2)-curves. Independently, we characterize spherical objects on blow-ups of minimal surfaces of positive Kodaira dimension.

1. Introduction

Let X be a smooth projective variety over the complex numbers and denote by $\mathsf{D}^b(X)$ the bounded derived category of coherent sheaves on X. If the canonical bundle ω_X is ample or anti-ample, then, by Bondal–Orlov [BO01], the group of autoequivalences $\mathrm{Aut}(\mathsf{D}^b(X))$ only consists of so-called *standard autoequivalences*, i.e.

$$\mathrm{Aut}(\mathsf{D}^b(X)) = \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X) \times \mathbb{Z}[1].$$

In general, the standard autoequivalences $\mathrm{Pic}(X) \rtimes \mathrm{Aut}(X) \times \mathbb{Z}[1]$ form a subgroup of $\mathrm{Aut}(\mathsf{D}^b(X))$ and $\mathsf{D}^b(X)$ often admits of non-standard autoequivalences, see, e.g., [Orl02] for the case of abelian surfaces and [BB17] for the case of $K3$ surfaces of Picard rank 1. A natural source for non-standard autoequivalences are so-called *spherical twists* [ST01].

In contrast to the case of varieties with trivial canonical class, a spherical object on a variety with nontrivial and non-torsion canonical class has to be supported on a proper closed subset, see Lemma 2.2. If X is a certain toric surface [BP14, Thm. 1, Thm. 2] or a surface of general type whose canonical model has at worst A_n-singularities [IU05, Thm. 1.5], then $\mathrm{Aut}(\mathsf{D}^b(X))$ is generated by standard equivalences and spherical twists.

In the first part of this paper, we focus on rational surfaces X which are blow-ups of $\mathbb{P}^2_{\mathbb{C}}$ in n very general points. By a result of de Fernex, recalled in Proposition 4.1, such a surface X does not contain any (-2)-curve. Motivated by the results of [IU05], it is reasonable to expect that the absence of (-2)-curves implies the absence of spherical objects. Indeed, utilizing the above result of de Fernex, we deduce the following Theorem 1.1 from the work of Uehara, see Section 4.

Theorem 1.1. *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in n very general points. Then the following statements hold:*

(i) *Any autoequivalence of X is standard, i.e.* $\mathrm{Aut}(\mathsf{D}^b(X)) = \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X) \times \mathbb{Z}[1]$.
(ii) *If Y is a smooth projective variety such that $\mathsf{D}^b(X) \cong \mathsf{D}^b(Y)$, then $X \cong Y$.*
(iii) *There exists no spherical object in $\mathsf{D}^b(X)$.*

By [Fav12, Cor. 4.4], for any smooth projective variety X we have that (i) implies (ii). Moreover, if X is a smooth projective variety of dimension ≥ 2 with nontrivial and non-torsion canonical class, then (i) implies (iii). Indeed, arguing as in Lemma 2.2, a spherical object S on such a variety X

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has to be supported on a proper closed subvariety. By [Huy06, Ex. 8.5 (ii)], the spherical twist T_S associated to S satisfies $T_S(S) = S[1 - \dim X]$ and $T_S(k(x)) = k(x)$ for any point $x \in X \setminus \mathrm{Supp}(S)$. Thus, T_S is a non-standard autoequivalence.

Note that Theorem 1.1 follows from [BO01] if $n \leq 8$ as X is a del Pezzo surface in this case. In fact, for $n \leq 8$ it is sufficient to require the blown up points to be in general position.

In Section 3 we give an elementary proof of Theorem 1.1 in the case $n = 9$ by arguing that for all integral curves $C \subseteq X$ the restriction $K_X|_C$ is nonzero in $\mathrm{CH}^1(C)_{\mathbb{Q}}$.

In Section 5 we consider blow-ups X of minimal surfaces Y of nonnegative Kodaira dimension. In contrast to the case of rational surfaces, (-2)-curves on X are strict transforms of (-2)-curves on Y, see Proposition 5.1. Thus, using [Ueh19], we obtain

Theorem 1.2 (Theorem 5.2)**.** *Let Y be a minimal surface of nonnegative Kodaira dimension and let X be the blow-up of Y in a nonempty finite set of points. Assume Y contains no (-2)-curves, e.g. Y has Kodaira dimension 1 and the elliptic fibration of Y has only irreducible fibers. Then $\mathsf{D}^b(X)$ admits only standard autoequivalences, i.e.*

$$\mathrm{Aut}(\mathsf{D}^b(X)) = \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X) \times \mathbb{Z}[1].$$

As outlined above, Theorem 1.2 implies that such X has no Fourier–Mukai partners and $\mathsf{D}^b(X)$ does not contain spherical objects.

In Proposition 5.7 we characterize spherical objects on blow-ups X of minimal surfaces Y of positive Kodaira dimension: An object in $\mathsf{D}^b(X)$ is spherical if and only if it is the pullback of a spherical object in $\mathsf{D}^b(Y)$ whose support is disjoint from the exceptional locus of $X \to Y$. If Y is a minimal surface of Kodaira dimension 1 whose elliptic fibration has only irreducible fibers, this characterization combined with the results of [Ueh16] gives an alternate proof that $\mathsf{D}^b(X)$ does not contain spherical objects, see Remark 5.9.

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Conventions. The term *surface* always refers to a smooth projective 2-dimensional variety over $\mathbb{C}$. For a variety X, we denote by $\mathrm{CH}^*(X)$ (resp. $\mathrm{CH}_*(X)$) the Chow groups of algebraic cycles modulo rational equivalence with integer coefficients graded by codimension (resp. dimension). We denote by $\mathrm{CH}^*(X)_{\mathbb{Q}} := \mathrm{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ the Chow groups with rational coefficients. A $(-k)$-curve C in a surface S is an integral smooth rational curve C with self-intersection number $-k$. The term "n general points in $\mathbb{P}^2_{\mathbb{C}}$" means that there exists a Zariski open subset $U \subseteq (\mathbb{P}^2_{\mathbb{C}})^n$ such that for any $(p_1, \ldots, p_n) \in U$ [...] holds. The term "n very general points in $\mathbb{P}^2_{\mathbb{C}}$" means that there exist countably many Zariski open subset $U_i \subseteq (\mathbb{P}^2_{\mathbb{C}})^n$ such that for any $(p_1, \ldots, p_n) \in \bigcap_i U_i$ [...] holds.

2. Preliminary Observations

Let X be a smooth projective variety. The support of an object $F \in \mathsf{D}^b(X)$ is by definition the closed subvariety

$$\mathrm{Supp}(F) := \bigcup_{i \in \mathbb{Z}} \mathrm{Supp}(\mathcal{H}^i(F)) \subseteq X$$

endowed with the unique reduced closed subscheme structure. If F is a simple object, i.e. $\mathrm{Hom}(F, F) = \mathbb{C}$, then $\mathrm{Supp}(F)$ is connected; see, e.g., [Huy06, Lem. 3.9].

Definition 2.1. An object $S \in \mathsf{D}^b(X)$ is called *spherical* if

$$\mathrm{Hom}(S, S[i]) = \begin{cases} \mathbb{C} & \text{if } i = 0, \dim X, \\ 0 & \text{else,} \end{cases}$$

and $S \otimes \omega_X \cong S$. An object $P \in \mathsf{D}^b(X)$ is called *point-like* if

$$\mathrm{Hom}(P, P[i]) = \begin{cases} 0 & \text{if } i < 0, \\ \mathbb{C} & \text{if } i = 0, \end{cases}$$

and $P \otimes \omega_X \cong P$.

Clearly, any shift of a skyscraper sheaf of a point $k(x)[m]$, $x \in X$, is a point-like object. By definition, a spherical object is a point-like object.

Denote by $p, q\colon X \times X \to X$ the projections and by $\Delta \hookrightarrow X \times X$ the diagonal embedding. If S is a spherical object on X, the object $\mathcal{P}_S \coloneqq \mathrm{Cone}(q^*S^\vee \otimes^{\mathrm{L}} p^*S \to \mathcal{O}_\Delta) \in \mathsf{D}^b(X \times X)$ is the Fourier–Mukai kernel of the *spherical twist* $T_S\colon \mathsf{D}^b(X) \to \mathsf{D}^b(X)$ given by $T_S(-) = \mathrm{R}p_*(\mathcal{P}_S \otimes^{\mathrm{L}} q^*(-))$. By [ST01, Thm. 1.2] a spherical twist is always an autoequivalence of $\mathsf{D}^b(X)$.

The condition $P \otimes \omega_X \cong P$ has the following consequence on the support of a point-like object:

Lemma 2.2. *Let X be a smooth projective positive dimensional variety with $K_X \neq 0$ in $\mathrm{CH}^*(X)_{\mathbb{Q}}$, i.e. ω_X is nontrivial and non-torsion. Then any point-like object $P \in \mathsf{D}^b(X)$ is supported on a connected proper closed subset.*

Moreover, if X is a surface, then either

(i) *$\mathrm{Supp}(P)$ is a point and $P \cong k(x)[m]$ for some $m \in \mathbb{Z}$, $x \in X$, or*

(ii) *$\mathrm{Supp}(P)$ is a, possibly reducible, connected curve $C = \bigcup_i C_i$ such that $K_X|_{\tilde{C}_i} = 0$ in $\mathrm{CH}^1(\tilde{C}_i)_{\mathbb{Q}}$, where C_i are the irreducible components of C and $\tilde{C}_i \to C_i$ are the normalizations.*

In particular, in (i) P is not spherical and in (ii) we have $K_X \cdot C = 0 \in \mathbb{Z}$.

Proof. Denote by $\mathcal{H}^i(P) \in \mathrm{Coh}\, X$ the i-th cohomology sheaf of P. Since ω_X is a line bundle, we have

$$\mathcal{H}^i(P) \otimes \omega_X = \mathcal{H}^i(P \otimes \omega_X) \cong \mathcal{H}^i(P),$$

which yields $\mathrm{ch}(\mathcal{H}^i(P))\,\mathrm{ch}(\omega_X) = \mathrm{ch}(\mathcal{H}^i(P))$ in $\mathrm{CH}^*(X)_{\mathbb{Q}}$. If $\mathcal{H}^i(P)$ had positive rank, then $\mathrm{ch}(\mathcal{H}^i(P))$ would be invertible in $\mathrm{CH}^*(X)_{\mathbb{Q}}$, hence $\mathrm{ch}(\omega_X) = 0$. This contradicts to K_X being non-torsion. Hence, all cohomology sheaves $\mathcal{H}^i(P)$ have rank zero and thus the generic point of X is not contained in the support of P. Thus, $\dim \mathrm{Supp}(P) < \dim X$ and $\mathrm{Supp}(P)$ is connected by [Huy06, Lem. 3.9].

Assume in addition that $\dim X = 2$. If P is supported on a point, then [Huy06, Lem. 4.5] shows that $P \cong k(x)[m]$ for some $x \in X$ and $m \in \mathbb{Z}$. In particular, $\chi(k(x)[m], k(x)[m]) = 0$, so P is not spherical. If $\mathrm{Supp}(P)$ is 1-dimensional, $\mathrm{Supp}(P)$ is a connected reduced, possibly reducible, curve.

Let $C_i \subseteq X$ be an irreducible curve, contained in $\mathrm{Supp}(P)$ and let $\tilde{C}_i \to C_i$ be its normalization. Denoting by $j\colon \tilde{C}_i \to C_i \hookrightarrow X$ composition, we obtain by the projection formula

$$K_X \cdot C_i = j_* j^* K_X \in \mathrm{CH}_0(X).$$

Let $\mathcal{H}$ be a cohomology sheaf of P which has nonzero rank restricted on C_i. The equality $\mathrm{ch}(\mathcal{H}) = \mathrm{ch}(\mathcal{H})\,\mathrm{ch}(\omega_X)$ on X shows $\mathrm{ch}(j^*\mathcal{H}) = \mathrm{ch}(j^*\mathcal{H})\,\mathrm{ch}(j^*\omega_X)$ on $\tilde{C}_i$. Since $j^*\mathcal{H}$ has nonzero rank, this implies that j^*K_X is torsion in $\mathrm{CH}_0(\tilde{C}_i)$. We conclude that the intersection number $K_X \cdot C_i = \deg(j_* j^* K_X)$ is zero. □

3. Blow-up in 9 Very General Points

Let $X \to \mathbb{P}^2_{\mathbb{C}}$ be the blow-up in the points $p_1, \dots, p_n \in \mathbb{P}^2$. We denote by $H \in \mathrm{Pic}(X)$ the class of the pullback of a hyperplane in $\mathbb{P}^2$ and by $E_i \in \mathrm{Pic}(X)$ the class of the (-1)-curve over the point p_i. Recall that the canonical class of X is given by $K_X = -3H + \sum_{i=1}^n E_i$ and that $\mathrm{Pic}(X)$ is freely generated by $H, E_1, \dots, E_n$. Moreover, the classes $H, E_1, \dots, E_n$ are pairwise orthogonal for the intersection pairing and satisfy $H^2 = 1$ and $E_i^2 = -1$ for $1 \leq i \leq n$. The first proof of Theorem 1.1 in the case of $n = 9$ points builds on the following observation:

Proposition 3.1. *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 9 very general points. Then $|-K_X|$ is zero-dimensional and the unique member is the strict transform $\tilde{C}$ of the unique smooth cubic curve $C \subseteq \mathbb{P}^2_{\mathbb{C}}$ which passes through the 9 points. Moreover, $K_X|_{\tilde{C}} \neq 0$ in $\mathrm{CH}^1(\tilde{C})_{\mathbb{Q}}$.*

Proof. Let $U \subseteq (\mathbb{P}^2_{\mathbb{C}})^9$ be the Zariski open subset parameterizing tuples of points $(p_1, \dots, p_9)$ such that $p_i \neq p_j$ for all $i \neq j$ and such that there is a unique smooth irreducible cubic $C \subseteq \mathbb{P}^2_{\mathbb{C}}$ passing through $p_1, \dots, p_9$. If $C \subseteq \mathbb{P}^2_{\mathbb{C}}$ is such a cubic, then $U \cap C^9$ is a nonempty Zariski open subset of C^9. Recall that for a fixed point $p_0 \in C$ the map $C \to \mathrm{CH}^1(C)$ sending $p \mapsto [p] - [p_0]$ is surjective. Thus, denoting H for class of the hyperplane in $\mathbb{P}^2_{\mathbb{C}}$, the map $C^9 \to \mathrm{CH}^1(C)$ sending

$(p_1,\dots,p_9)\mapsto 3H|_C-[p_1]-\dots-[p_9]$ is surjective. Since $\mathrm{Pic}^0(C)$ has only countably many torsion points, the set

$$Z := \left\{(p_1,\dots,p_9)\in U \mid 3H|_C-\sum_{i=1}^9[p_i]=0 \text{ in } \mathrm{CH}^1(C)_{\mathbb{Q}} \text{ where } p_1,\dots,p_9\in C \text{ and } C\in|3H|\right\}$$

is a proper subset of U. To conclude the lemma, it is enough to show that Z is a countable union of closed algebraic subsets of U. For this, we rewrite Z as follows: Let

$$\mathcal{C} := \left\{(p_1,\dots,p_9,f,p)\in U\times\mathbb{P}(H^0(\mathbb{P}^2_{\mathbb{C}},\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3)))\times\mathbb{P}^2_{\mathbb{C}} \mid f(p_1)=\dots=f(p_9)=f(p)=0\right\}\xrightarrow{\pi}U$$
$$(p_1,\dots,p_9,f,p)\mapsto(p_1,\dots,p_9)$$

be the family of cubic curves through the 9 points $(p_1,\dots,p_9)$. The map π admits sections $s_i\colon U\to\mathcal{C}$ given by $(p_1,\dots,p_9)\mapsto(p_1,\dots,p_9,f,p_i)$ where f is the unique cubic polynomial vanishing at $p_1,\dots,p_9$. We obtain cycles $\mathcal{P}_i := [s_i(U)]\in\mathrm{CH}^1(\mathcal{C})$ such that the restriction $\mathcal{P}_i|_{C_{\bar p}}$ to the fiber $C_{\bar p}=\pi^{-1}(\bar p)$ over a point $\bar p=(p_1,\dots,p_9)$ is $[p_i]\in\mathrm{CH}^1(C_{\bar p})$. Let $\mathcal{H}\in\mathrm{CH}^1(\mathcal{C})$ be the pullback of the hyperplane class on $\mathbb{P}^2_{\mathbb{C}}$. Then $3\mathcal{H}-\sum_i\mathcal{P}_i$ is a cycle in $\mathrm{CH}^1(\mathcal{C})$ and the set Z is the locus of points $\bar p\in U$ such that $(3\mathcal{H}-\sum_i\mathcal{P}_i)|_{C_{\bar p}}$ vanishes in $\mathrm{CH}^1(C_{\bar p})_{\mathbb{Q}}$. By [Voi14, Lem. 3.2], Z is a countable union of closed algebraic subvarieties. □

Lemma 3.2. *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in $n\le 8$ general points or $n=9$ very general points. Then for any integral curve $C\subseteq\mathbb{P}^2_{\mathbb{C}}$, $K_X|_{\tilde C}\ne 0$ in $\mathrm{CH}^1(\tilde C)_{\mathbb{Q}}$, where $\tilde C$ is the normalization of the strict transform of C.*

Proof. If $n\le 8$, then $-K_X$ is ample and thus $-K_X\cdot\tilde C>0$ for every integral curve $\tilde C\subseteq X$. In particular, $K_X|_{\tilde C}\ne 0$ since $\deg(-K_X|_{\tilde C})=-K_X\cdot\tilde C>0$.

If $n=9$, then $K_X^2=0$. It follows from the Hodge index theorem that the lattice $K_X^\perp := \{v\in\mathrm{Pic}(X)\mid K_X\cdot v=0\}$ is negative semi-definite. Furthermore, any class $v\in K_X^\perp$ with $v^2=0$ is of the form $v=mK_X$ for some $m\in\mathbb{Z}$. For $m>0$, mK_X is not effective and for $m<0$ the linear system $|mK_X|$ is zero-dimensional, see, e.g., [CM00, Thm. 2.5]. Its unique member is the (multiple of the) strict transform of the unique smooth cubic C through the 9 blown up points. We conclude by Proposition 3.1 that $K_X|_{\tilde C}\ne 0$ in $\mathrm{CH}^1(\tilde C)_{\mathbb{Q}}$. □

With the above Lemma 3.2, we can prove Theorem 1.1 in case of $n=9$ points as follows:

Proof of Theorem 1.1 (iii) for $n\le 9$ points. Assume for contradiction that $S\in\mathsf{D}^b(X)$ is a spherical object. By Lemma 2.2, S is supported on a connected curve $C=\bigcup_i C_i$ such that $K_X|_{C_i}=0\in\mathrm{CH}^1(C_i)_{\mathbb{Q}}$ for all irreducible components C_i of C. By Lemma 3.2 such curves C_i do not exist. □

Proof of Theorem 1.1 (i) and (ii) for $n\le 9$ points. Let $\phi\colon\mathsf{D}^b(Y)\to\mathsf{D}^b(X)$ be an equivalence. For any point $y\in Y$ the skyscraper sheaf $k(y)$ satisfies $k(y)\otimes\omega_Y\cong k(y)$ and thus $\phi(k(y))\otimes\omega_X\cong\phi(k(y))$. Since $\mathrm{Hom}(k(y),k(y)[i])=\mathrm{Hom}(\phi(k(y)),\phi(k(y))[i])$ for all $i\in\mathbb{Z}$, $\phi(k(x))$ is a point-like object. By Lemma 2.2, $\mathrm{Supp}(\phi(k(y)))$ is either a point or $\mathrm{Supp}(\phi(k(y)))=\bigcup_i C_i$, where each C_i is an integral curve with $K_X|_{C_i}=0\in\mathrm{CH}^1(C_i)_{\mathbb{Q}}$. By Lemma 3.2 such curves C_i do not exist. Hence, $\phi(k(y))$ is supported on a point $x\in X$ and $\phi(k(y))=k(x)[m]$ for some $m\in\mathbb{Z}$. Moreover, by [Huy06, Cor. 6.14] the locus of $y\in Y$ such that $\phi\circ[-m](k(y))$ is a skyscraper sheaf is open. Since Y is connected, this locus is the whole of Y, which shows that the shift m in $\phi(k(y))=k(x)[m]$ is independent of $y\in Y$. Thus $\phi\circ[-m]$ sends skyscraper sheaves to skyscraper sheaves and [BM17, § 3.3] (or [Huy06, Cor. 5.23]) shows that $\phi\circ[-m]=f_*(\mathcal{L}\otimes -)$ for some line bundle $\mathcal{L}\in\mathrm{Pic}(Y)$ and isomorphism $f\colon Y\to X$. This proves (ii) and shows that in the case $Y=X$ the autoequivalence ϕ is a standard autoequivalence. Thus, (i) follows. □

Remark 3.3. If one could prove the conclusion of Lemma 3.2 for blow-ups X of $\mathbb{P}^2_{\mathbb{C}}$ in $n>9$ very general points, then Theorem 1.1 would follow by the same arguments as used above in the case of $n\le 9$ points.

4. Proof of the General Case

In the general case of Theorem 1.1 (i) and (ii) follow from the work of Uehara [Ueh19, Thm. 1.1, Thm. 1.3], Kawamata [Kaw02], and the following result of de Fernex:

Proposition 4.1 ([Fer05, Prop. 2.4])**.** *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of points in very general position. If $C \subseteq X$ is an integral rational curve with $C^2 < 0$, then C is a (-1)-curve, that is a smooth rational curve of self-intersection -1.*

Proof of Theorem 1.1 (i) and (ii). Recall, e.g. from [CD12, Prop. 2.2], that if Y is a rational surface admitting a minimal elliptic fibration, then Y can be obtained from $\mathbb{P}^2_{\mathbb{C}}$ by blowing up 9, possibly infinitely near, points and, for some $m > 0$, the linear system $|-mK_Y|$ is a pencil. Hence, if X is the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of points in very general position, then X admits no minimal elliptic fibration. Indeed, this is clear if the number of blown up points is different from 9. In the case of 9 blown up points the linear system $|-mK_X|$ is zero-dimensional for any $m > 0$, so it is not a pencil. By [Kaw02, Thm. 1.6], a non-minimal surface admits nontrivial Fourier–Mukai partners only if it admits a minimal elliptic fibration. Hence, Theorem 1.1 (ii) follows.

Let Y be any surface and let $\Phi_P \colon \mathsf{D}^b(Y) \to \mathsf{D}^b(Y)$ be an autoequivalence with Fourier–Mukai kernel $P \in \mathsf{D}^b(Y \times Y)$. We denote by $\mathrm{Comp}(\Phi_P)$ the set of irreducible components of $\mathrm{Supp}(P) \hookrightarrow Y \times Y$ and by

$$N_Y := \max\{\dim W \mid W \in \mathrm{Comp}(\Phi_P), \Phi_P \in \mathrm{Aut}(\mathsf{D}^b(Y))\}$$

the *Fourier–Mukai support dimension* of Y. By Uehara's classification [Ueh19, Thm. 1.1], the equality $N_Y = 2$ is equivalent to Y admitting no minimal elliptic fibration and K_Y being not numerically equivalent to zero. Hence, for X the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of points in very general position we have $N_X = 2$.

If Y is a surface with $N_Y = 2$ such that the union of all (-2)-curves in Y forms a disjoint union of configurations of type A, then, by [Ueh19, Thm. 1.3], $\mathrm{Aut}(\mathsf{D}^b(Y))$ is generated by standard autoequivalences and spherical twists. For X the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of points in very general position, de Fernex' Proposition 4.1 shows that X contains no (-2)-curve. Hence, Theorem 1.1 (i) follows. □

5. Surfaces of Nonnegative Kodaira Dimension

5.1. **Autoequivalences.** In contrast to the case of negative Kodaira dimension, blowing up points in arbitrary position on minimal surfaces of nonnegative Kodaira dimension does not give rise to new (-2)-curves.

Proposition 5.1. *Let Y be a minimal surface of nonnegative Kodaira dimension and let $p \colon X \to Y$ be the blow-up of Y in a set of points $p_1, \dots, p_n \in Y$. Then every (-2)-curve C in X is the strict transform of a (-2)-curve C_0 in Y such that $p_i \notin C_0$ for $1 \le i \le n$.*

Proof. We denote by E_i the exceptional divisor over the i-th blown up point p_i for $1 \le i \le n$. Let $C \subseteq X$ be a (-2)-curve. By adjunction, we have

$$0 = g(C) = 1 + \frac{1}{2}(C^2 + C \cdot K_X),$$

where $g(C)$ denotes the geometric genus of C. Thus, $C \cdot K_X = 0$. Further, since C is not one of the exceptional curves E_i, C is the strict transform of a curve $C_0 \subseteq Y$. We have

$$0 = C \cdot K_X = C_0 \cdot K_Y + \sum_{i=1}^{n} m_i,$$

where m_i is the multiplicity of C_0 at p_i. Since K_Y is nef, each of the m_i is zero, in other words $p_i \notin C_0$ for $1 \le i \le n$. We conclude that C_0 is a smooth rational curve with $K_Y \cdot C_0 = 0$, hence, by adjunction, a (-2)-curve. □

As a consequence of Proposition 5.1 and [Ueh19], we obtain the following

Theorem 5.2. *Let Y be a minimal surface of nonnegative Kodaira dimension and let X be the blow-up of Y in a nonempty finite set of points. Assume Y contains no (-2)-curves, e.g. Y has Kodaira dimension 1 and the elliptic fibration of Y has only irreducible fibers. Then $\mathsf{D}^b(X)$ admits only standard autoequivalences, i.e.*

$$\mathrm{Aut}(\mathsf{D}^b(X)) = \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X) \times \mathbb{Z}[1].$$

Proof. By Proposition 5.1, X contains no (-2)-curves. Thus, the statement follows from [Ueh19, Thm. 1.1, Thm. 1.3] if X admits no minimal elliptic fibration. The latter can be shown as follows: Recall, e.g. from [CDL23, Cor. 4.1.7], that a surface S with minimal elliptic fibration satisfies $K_S^2 = 0$. If $\kappa(Y) = 0$, then K_Y is numerically equivalent to zero. Hence, $K_Y^2 = 0$ and therefore $K_X^2 < 0$. If $\kappa(Y) = 1$, then Y has an elliptic fibration and therefore $K_Y^2 = 0$. Hence, $K_X^2 < 0$. Finally, if $\kappa(Y) = 2$, then X has no elliptic fibration by [Bar+04, Ch. V, Prop. 12.5]. □

Remark 5.3. Note that the description of autoequivalences as in Theorem 5.2 is not true for a minimal surface Y. For example, if $\kappa(Y) = 1$, then $\mathrm{Aut}(\mathsf{D}^b(Y))$ can be characterized as in [Ueh16, Thm. 4.1]. In that case, as outlined in the proof of [Ueh19, Thm. 1.1], Y admits an autoequivalence $\Phi_{\mathcal{U}}$ where $\mathcal{U}$ is the universal sheaf on $Y \times J_Y(1,1)$ and $J_Y(1,1) \cong Y$ is a moduli space of stable sheaves on a smooth fiber of the elliptic fibration of Y. In this case, the support of $\mathcal{U}$ is 3-dimensional, thus $\Phi_{\mathcal{U}}$ does not lift to an autoequivalence of a blow-up of Y.

Remark 5.4 (Infinitely near points). Let X be a non-minimal surface of nonnegative Kodaira dimension with minimal model Y. If the (-2)-curves in Y only form chains of type A, then it is possible to describe $\mathrm{Aut}(\mathsf{D}^b(X))$ as in [Ueh19, Thm. 1.3]. Indeed, arguing as in [IU05, Thm. 1.5] one shows that the (-2)-curves in X only form chains of type A. Thus, [Ueh19, Thm. 1.3] applies and shows that $\mathrm{Aut}(\mathsf{D}^b(X))$ is generated by standard autoequivalences and spherical twists.

5.2. **Spherical Objects.** Similar to Proposition 5.1, spherical objects in the blow-up of a minimal surface of positive Kodaira dimension are completely determined by the minimal surface.

We begin with recalling two elementary Lemmata 5.5 and 5.6 regarding morphisms and the support of complexes of sheaves. As we were unable to find a suitable statement in the literature, we include a proof of Lemma 5.5.

Lemma 5.5. *Let X be a smooth projective variety and let $F, G \in \mathsf{D}^b(X)$.*

(i) *If $\mathrm{Supp}(F) \cap \mathrm{Supp}(G) = \emptyset$, then $\mathrm{Hom}_{\mathsf{D}^b(X)}(F, G) = 0$.*
(ii) *If $D \subseteq X$ is a divisor and $\mathrm{Supp}(F) \cap D = \emptyset$, then $F \otimes \mathcal{O}_X(D) = F$.*

Proof. We first prove (i). The condition $\mathrm{Supp}(F) \cap \mathrm{Supp}(G) = \emptyset$ implies $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{H}^{-q}(F), \mathcal{H}^l(G)) = 0$ for all $p, q, l \in \mathbb{Z}$. Recall, e.g. from [Huy06, p. 77], that we have a spectral sequence

$$E_2^{p,q} = \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{H}^{-q}(F), \mathcal{H}^l(G)) \Rightarrow \mathcal{E}xt^{p+q}_{\mathcal{O}_X}(F, \mathcal{H}^l(G))$$

for every $l \in \mathbb{Z}$. Similarly, we have a spectral sequence

$$E_2^{p,q} = \mathcal{E}xt^p_{\mathcal{O}_X}(F, \mathcal{H}^q(G)) \Rightarrow \mathcal{E}xt^{p+q}_{\mathcal{O}_X}(F, G).$$

Thus, $\mathrm{Supp}(F) \cap \mathrm{Supp}(G) = \emptyset$ implies $\mathcal{E}xt^l_{\mathcal{O}_X}(F, G) = 0$ for all $l \in \mathbb{Z}$. Finally, the local-to-global spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(F, G)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(F, G)$$

shows $\mathrm{Ext}^{p+q}_{\mathcal{O}_X}(F, G) = 0$.

To prove (ii), assume that $D \subseteq X$ is a divisor and that $\mathrm{Supp}(F) \cap D = \emptyset$. The ideal sheaf sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

yields an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_D, F) \to F \to F \otimes \mathcal{O}_X(D) \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_D, F) \to 0.$$

As argued above, we have $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_D, F) = 0 = \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_D, F)$. Hence, $F \to F \otimes \mathcal{O}_X(D)$ is an isomorphism. □

Lemma 5.6 ([BM02, Lem. 5.3])**.** *Let X be a smooth projective variety and $F \in \mathsf{D}^b(X)$. Then a point $x \in X$ lies in* $\mathrm{Supp}(F)$ *if and only if* $\mathrm{Hom}_{\mathsf{D}^b(X)}(F, k(x)[l]) \neq 0$ *for some $l \in \mathbb{Z}$.*

The following Proposition 5.7 characterizes spherical objects in blow-ups of minimal surfaces of positive Kodaira dimension.

Proposition 5.7. *Let Y be a minimal surface of positive Kodaira dimension and let $p\colon X \to Y$ be the blow-up of Y in a set of points $p_1, \dots, p_n \in Y$. Then every spherical object in $\mathsf{D}^b(X)$ is of the form $\mathrm{L}p^*S$ for some spherical object $S \in \mathsf{D}^b(Y)$. Moreover, if $S \in \mathsf{D}^b(Y)$ is spherical, then $\mathrm{L}p^*S$ is spherical if and only if $p_i \notin \mathrm{Supp}(S)$ for all $1 \leq i \leq n$.*

Proof. We denote by E_i the exceptional divisor over the i-th blown up point p_i for $1 \leq i \leq n$. We first prove the following

Claim. If $S' \in \mathsf{D}^b(X)$ is a spherical object, then $\mathrm{Supp}(S')$ is disjoint from each E_i.

Proof of the claim. Assume $S' \in \mathsf{D}^b(X)$ is spherical, then, by Lemma 2.2, $\mathrm{Supp}(S') = \bigcup_i C_i$, where each C_i is an integral curve with $K_X \cdot C_i = 0$. Since $K_X = p^*K_Y + \sum_i E_i$, such curve C_i is the strict transform of a curve in Y. Moreover, if C_0 is a curve in Y, the strict transform of C_0 has class $p^*C_0 - \sum_i m_i E_i$, where m_i is the multiplicity of C_0 at p_i. We compute that

$$K_X \cdot \left(p^*C_0 - \sum_{i=1}^n m_i E_i \right) = K_Y \cdot C_0 + \sum_{i=1}^n m_i.$$

Since K_Y is nef, we have $K_Y \cdot C_0 \geq 0$ and therefore $p_i \notin C_0$ for all $1 \leq i \leq n$. □

Recall that $\mathsf{D}^b(X)$ admits a semiorthogonal decomposition

$$\mathsf{D}^b(X) = \langle \mathcal{O}_{E_1}(-1), \dots, \mathcal{O}_{E_n}(-1), \mathrm{L}p^*\mathsf{D}^b(Y) \rangle.$$

Since $\mathrm{Supp}(S')$ is disjoint from each E_i, we have, by Lemma 5.5,

$$\mathrm{Hom}_{\mathsf{D}^b(X)}(S', \mathcal{O}_{E_i}(-1)[l]) = 0 = \mathrm{Hom}_{\mathsf{D}^b(X)}(\mathcal{O}_{E_i}(-1), S'[l])$$

for every $l \in \mathbb{Z}$. Hence, $S' \in \mathrm{L}p^*\mathsf{D}^b(Y)$, i.e., there exists a object $S \in \mathsf{D}^b(Y)$ such that $\mathrm{L}p^*S \cong S'$. Note that $\mathrm{R}p_*\mathcal{O}_X = \mathcal{O}_Y$ implies

$$\begin{aligned} \text{(5.8)} \qquad \mathrm{Hom}_{\mathsf{D}^b(X)}(S', S'[l]) &= \mathrm{Hom}_{\mathsf{D}^b(X)}(\mathrm{L}p^*S, \mathrm{L}p^*S[l]) = \mathrm{Hom}_{\mathsf{D}^b(Y)}(S, \mathrm{R}p_*\mathrm{L}p^*S[l]) \\ &= \mathrm{Hom}_{\mathsf{D}^b(Y)}(S, S \otimes^{\mathrm{L}} \mathrm{R}p_*\mathcal{O}_X[l]) = \mathrm{Hom}_{\mathsf{D}^b(Y)}(S, S[l]). \end{aligned}$$

for every $l \in \mathbb{Z}$. Moreover, since $\mathrm{Supp}(S')$ is disjoint from the exceptional divisors E_i, Lemma 5.5 shows that $\mathrm{L}p^*S \otimes \mathcal{O}_X(\sum_i E_i) = \mathrm{L}p^*S$. Hence, $\mathrm{L}p^*S \otimes p^*\omega_Y \cong \mathrm{L}p^*S$. Pushing forward via $\mathrm{R}p_*$ and using the projection formula shows that $S \otimes \omega_Y \cong S$. Thus, S is a spherical object in $\mathsf{D}^b(Y)$.

Now let $S \in \mathsf{D}^b(Y)$ be a spherical object. As in (5.8), we have

$$\mathrm{Hom}_{\mathsf{D}^b(X)}(\mathrm{L}p^*S, \mathrm{L}p^*S[l]) = \mathrm{Hom}_{\mathsf{D}^b(Y)}(S, S[l])$$

for every $l \in \mathbb{Z}$. Thus, $\mathrm{L}p^*S$ is spherical if $\mathrm{L}p^*S \otimes \omega_X \cong \mathrm{L}p^*S$. Let $x \in X$ be a point, then $\mathrm{R}p_*k(x) = k(p(x))$ and by adjunction

$$\mathrm{Hom}_{\mathsf{D}^b(X)}(S, \mathrm{R}p_*k(x)[l]) = \mathrm{Hom}_{\mathsf{D}^b(X)}(\mathrm{L}p^*S, k(x)[l])$$

for every $l \in \mathbb{Z}$. Hence, Lemma 5.6 shows that $\mathrm{Supp}(\mathrm{L}p^*S) = p^{-1}(\mathrm{Supp}(S))$. By the previous claim, it is necessary that $p^{-1}(\mathrm{Supp}(S))$ is disjoint from each E_i for $\mathrm{L}p^*S$ to be spherical. On the other hand, this is also sufficient, since $\mathrm{L}p^*S \otimes \mathcal{O}_X(\sum_i E_i) = \mathrm{L}p^*S$ holds by Lemma 5.5 if $p^{-1}(\mathrm{Supp}(S))$ is disjoint from each E_i. □

Remark 5.9. Let Y be a minimal surface of Kodaira dimension 1 whose elliptic fibration has only irreducible fibers. It follows from the description of $\mathrm{Aut}(\mathsf{D}^b(Y))$ in [Ueh16] that $\mathsf{D}^b(Y)$ does not contain spherical objects. Thus, if X is a blow-up of Y in a finite set of points, then, by Proposition 5.7, $\mathsf{D}^b(X)$ does not contain spherical objects either. Alternately, this can also be deduced from Theorem 5.2.

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Fakultät für Mathematik, Technische Universität München, D-85747 Garching bei München, Germany
Email address: xianyu.hu@tum.de

Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany
Email address: jkrah@math.uni-bielefeld.de